UDC 517.9

PERIOD DOUBLING BIFURCATIONS IN A SIMPLE MODEL OF A DISTRIBUTED SYSTEM+

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1. As known, many dynamical systems of different physical nature demonstrate either regular, periodic, or irregular, chaotic behavior in dependence on some parameters [1]. One of the roads to the onset of chaos was discovered by Feigenbaum [2] in a study of a simple model

$$x_{m+1} = f_\lambda(x_m). \tag{1}$$

Here x_m characterizes the system state, m is discrete time, λ is parameter, $f_\lambda(x)$ is a function possessing a quadratic maximum, say $f_\lambda(x) = \lambda(1-x^2)$. The system (1) has a critical point of transition from order to chaos at $\lambda = \lambda_{cr}^0$, in a vicinity of it the behavior of generated sequences x_m obeys a universal scaling law non-dependent on the concrete form of the function [2]. While approaching λ to from below, the period-doubling bifurcations take place: at a 2^n-periodic orbit (cycle) loses stability, and a stable cycle arises. At the critical point the closest to the extremum of elements of the 2^n- and 2^{n+1}-cycles are related as $x(2^n) \cong ax(2^{n+1})$. The constants $a = -2.5029$ and $\delta = 4.6692$ are universal while λ_{cr}^0 and A depend on the function $f_\lambda(x)$.* Hence, it is convenient sometimes to use parameter, for which the bifurcational values are universal ($\Lambda_n^0 = -\delta^{-n}$). The mentioned regularities were observed in numerical computations and in experiments for a number of systems with discrete as well as with continuous time [2-5]. The next important step should consist in derivation of results analogous by their nature for distributed systems. In this work we study from this point of view a model of delayed feedback generator interesting for electronic applications [6-8].

Let us consider a closed into a ring chain consisting of an inertia-less amplifier, an inertial linear element – filter, and a delay line [6]. Let us accept that the signal generated is determined by one time-dependent variable $x(t)$, and the dynamical equation for the model is

$$x(t+T) = f_\lambda(\hat{l}x(t)). \tag{2}$$

Here T is the delay time, $\hat{l}$ is linear operator with a spectral transfer function that has a form of $e^{-i\omega t}\hat{l}e^{i\omega t} = l(\omega) \approx 1-\omega^2\tau^2$, $\tau << T$ in a low-frequency band. The function $f_\lambda(x)$ represents the nonlinear properties of the amplifier, and parameter λ charicterizes the depth of the feedback.

If the dependence $x(t)$ is slow, then from (2) it follows that $x(t+T) = f_\lambda(x(t))$, and analysis of the dynamics in terms of this equation is reduced to consideration of the sequences (1). We will show that the model (2) as well demonstrates an infinite sequence of period-doubling bifurcations and find bifurcation curves in the parameter plane ($\Lambda, T/\tau$). The hypotheses of scaling will be formulated, and their verification in numerical experiment will be performed for a concretized model (2) with time discretized in the scale of τ:

$$x_{k+K} = \lambda(1-(\hat{l}x_k)^2), \quad \hat{l}x_k = x_k + \alpha(x_{k-1} - 2x_k + x_{k+1}). \tag{3}$$

2. Figure 1 presents typical dependencies $x(t)$ of period $2T$, $4T$, $8T$ obtained numerically for the model (3). As it is seen from the figure, the signal contains more ore less long

+ Russian original: *Izvestja VUZov – Radiofizika, 1982, vol.25, No 11, pp.1364-1368.*

* For $f_\lambda(x) = \lambda(1-x^2)$ the constants are $\lambda_{cr}^0 = 1.1837$ and $A = 1.4291$.

plateaus, where a value of $x(t)$ is close to one of the elements of the cycle of the map (1), and regions of fast change of amplitude, the domain walls between the neighboring plateaus. The transformation of the wall structure in a course of a period-doubling bifurcation consists in appearance of tails to the left and to the right from the wall, which ensure values of x corresponding to the new (doubled) cycle at the middle of the plateaus. The first hypothesis of scaling consists in the following: the tails, which have been arisen after bifurcation of different order n are similar, their characteristic time scale grows as $2^{n/2}$, and their scale in x direction decreases proportionally to the distance between the elements of the 2^n- and 2^{n+1}-cycles of the map (1). Quantitative verification of this statement was performed for the tails associated with the similar in Feigenbaum's sense elements of the periodic orbits at the critical point $\lambda = \lambda^0_{cr}$. Configuration of the tail is defined as a difference of the functions $x(t)$ corresponding to (unstable) 2^n- and 2^{n+1} –periodic regimes, and it is depicted in Fig.2 in the normalized coordinates $\xi = (t/\tau)2^{-n/2}$, $\eta = h_n a^n$.** Observe that already for $n \geq 2$ the tail forms are well described by a universal curve. Coincidence of the plots h_n at $\xi \to \infty$ implies validity of the scaling law [2], but for finite ξ it is a novel relevant result.

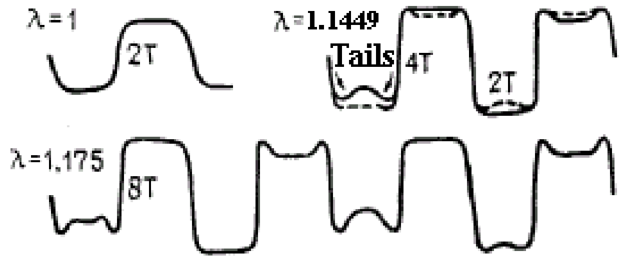

Fig.1

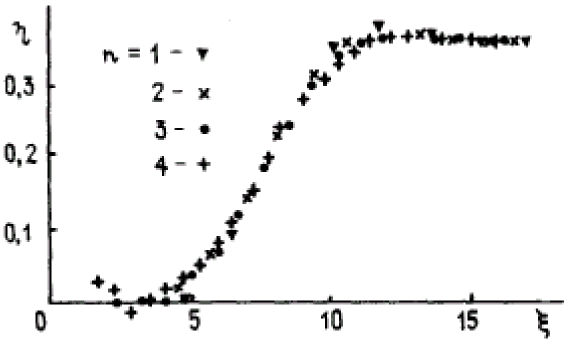

Fig.2

3. In accordance with the stated scaling law, length of a tail arisen after n-th bifurcation is of order $2^{n/2}\tau$. Hence, the neighboring walls separated by an interval T do not "feel" each other while $2^{n/2}\tau << T$. It means that the period-doubling bifurcations of order $n << N \approx 2\log_2(T/\tau)$ occur in the model (2) approximately at the same parameter values as in the model (1): $\lambda_n \approx \lambda^0_n$, $\Lambda_n \approx -\delta^{-n}$. The difference of λ_n from λ^0_n at $n \geq N$ appears due to interaction of the neighboring walls by means of their tails. As the tails obey the scaling law, the same should be true for their interaction as well. So, we expect that the second scaling hypothesis is valid:

$$\Lambda_n = \delta^{-n}\varphi(\vartheta 2^{-n/2}). \tag{4}$$

** The origin for time is placed at the scaling center of the set of tails from the condition of the best coincidence of the plots for h_4 and h_3 and is located at the distance of order of τ from the wall center.

Here φ is a universal function, $\vartheta = T/\tau + \Delta$ is a normalized interval between scaling centers of the interacting tails, Δ is a dimensionless constant of order of several units (non universal). The condition of application of the formula (4) is the inequality $T >> \tau$. Indeed, only in this case the tails becomes universal before appearance of essential interaction between them.

4. The numerically found bifurcation values λ_n for the model (3) at $\alpha = 1/3$ are given in the Table. To simplify the calculations we account an evident from Fig.1 approximate symmetry of the signal (the relation used see in the title of the Table). Parameters of the model M and α determine the ratio $T/\tau = (2M-1)/\sqrt{\alpha}$.

Table

Bifurcation values of the control parameter λ for the model

$$x_m^{n+1} = \lambda\,[1-(y_m^n)^2],\quad y_m^n = x_m^n + \alpha\,(x_{m-1}^n - 2x_m^n + x_{m+1}^n),$$

$$x_{-1/2}^n = x_{1/2}^n,\quad x_{M+1/2}^n = x_{M-3/2}^{n+1}$$

	M = 5	6	7	8	9	10		n
λ_n	0,890128	0,882013	0,877413	0,874549	0,872647	0,871250	0,866025	1
	1,136033	1,128496	1,124846	1,122819	1,121578	1,120763	1,118034	2 λ_n^0
	1,191035	1,180209	1,175386	1,173237	1,172094	1,171419	1,169658	3
	1,2029[illegible]1	1,191980	1,187312	1,184826	1,183212	1,182289	1,180697	4
	1,205470	1,194552	1,189882	1,187344	1,185727	1,184835	1,183060	5
						1,185386	1,183566	6
λ_{cr}	1,206171	1,195255	1,190582	1,188040	1,186428	1,185536	1,183704	λ_{cr}^0
λ_{cr} (5)	1,205810	1,195583	1,190592	1,187943	1,186441	1,185541		

Let us perform verification of the scaling hypothesis (4) in two stages.

1) Scaling for the line of critical points. As seen from the Table, for $T/\tau = \text{const}$ the values λ_n converge to a finite limit. The relation (4) may be agreed with this result if we assume that $\varphi(\varsigma)\big|_{\varsigma\to 0} = \kappa\varsigma^{-\nu}$, with $\nu = 2\log_2\delta = 4.4463$. Setting $n \to \infty$ in (4), we obtain an equation for the critical line in the plane (Λ, θ):

$$\Lambda_{cr} = \kappa\theta^{-\nu},\ \text{or}\ (\lambda_{cr} - \lambda_{cr}^0)/A = \kappa(T/\tau + \Delta)^{-\nu} \qquad (5)$$

The constants $\kappa \approx 17910$ (universal) and $\Delta \approx 7.518$ were determined from the obtained set of λ_{cr} by the least square method and used for computation of λ_{cr} from (5) (the last row of the Table). Good coincidence with the numerical data is evident.

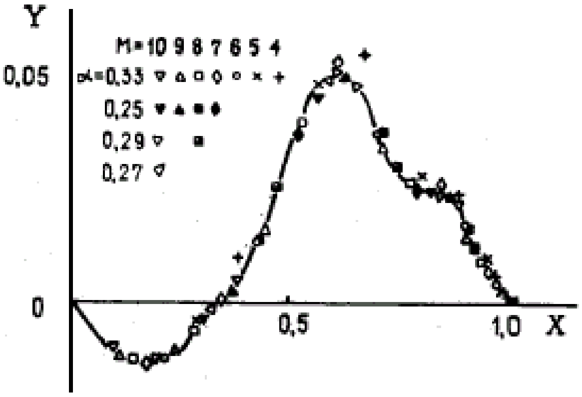

Fig.3

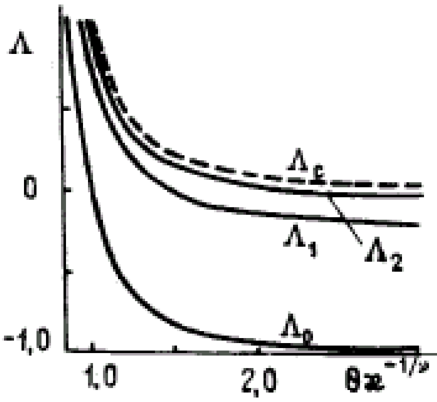

Fig.4

2) Scaling for the period-doubling bifurcations. The scaling relation (4) may be presented in the form

$$Y = \Phi(X),\ \ X = \left(\frac{\lambda_{cr} - \lambda^0_{cr}}{\lambda_{cr} - \lambda^0_n}\right)^{2/\theta},\ \ Y = \frac{\lambda_{cr} - \lambda_n - \lambda^0_{cr} + \lambda^0_n}{\lambda_{cr} - \lambda^0_n}. \qquad (6)$$

In Fig.3 we present numerical data in the coordinates () relating to bifurcations of different order n for several values of M and α. The data corresponding to $T/\tau >> 1$ fit well onto the same curve, and it confirms the relation (4). As expected, decreasing T/τ one observes deflections from the scaling law.

In Fig.4 the bifurcation lines are shown on the plane $(\Lambda,\ \theta)$ plotted with a use of the function Φ (see Fig.3) and the relations (7). Note that the Feigenbaum law $\Lambda_{cr} - \Lambda_n \propto \delta^{-n}$ works for our system only in a close neighborhood of the critical line:. Nontrivial form of the curve $\Phi(X)$ tells us that in the domain $|\Lambda_{cr} - \Lambda| \cong \Lambda_{cr}$ the bifurcation lines obey more complex law, but it also does not depend on concrete form of the function $f_\lambda(x)$ and of the filter characteristic $l(\omega)$.

The developed approach may be useful in application to various distributed system if the corresponding local (point) system undergoes the transition from order to chaos via the period-doubling bifurcations.

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