Critical behavior of one-dimensional chains

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1. A standard approach in the theory of oscillations and waves is to first analyze a simple oscillatory system and then use the results to analyze a chain of coupled systems. This approach makes it possible to make the transition from elementary systems to systems with a large number of degrees of freedom. It was recently learned[1,2] that many simple dynamic systems may undergo transitions from a periodic oscillation regime to a stochastic regime through a bifurcation – a doubling of the period. The sequence of bifurcation values of the parameter converges toward a finite limit: the critical point. It is natural to ask about the behavior of a chain of coupled systems exhibiting such properties. In the present letter we analyze the changes in the dynamics of small perturbations of a spatially uniform solution in chains of this sort upon a repeated doubling of the temporal period. For simplicity we consider the behavior only at the critical point. The results may prove useful for reaching an understanding of the general principles involved in the onset of a stochastic regime in a distributed system.

2. As the elementary cell from which we will construct the chain we adopt a dynamic system describable by the equation

$$x' = g(x), \tag{1}$$

where x is a variable characterizing the state of the cell at the discrete time n, x' is the corresponding variable for the time n + 1, and g(x) is the Feigenbaum function.[1] A small perturbation ξ of the variable x transforms over a unit time in accordance with $\xi' = g'(x)\xi$.

Turning now to a one-dimensional chain of coupled cells, we assume that Eq. (1) describes a spatially uniform solution. We consider a small perturbation of the type $\xi e^{i\beta m}$ (β is the wave number, and m is the cell index) of this uniform solution. To describe the evolution of the perturbation we incorporate a term linear in ξ in the equation for the perturbation:

$$\xi' = g'(x)\xi + \mathcal{E}\psi(x,\beta)\xi. \tag{2}$$

Here $\mathcal{E}$ is a small parameter, and $\psi(x, \beta)$ is a function whose form depends on the particular way in which the coupling between cells is introduced [$\psi(x, 0) = 0$].

We impose a renormalization (doubling) transformation on Eqs. (1) and (2) (Ref. 1): We express x and ξ at the time n + 2 in terms of their values at the time n, ignoring the term of order $\mathcal{E}^2$,

$$x'' = g(g(x)),\quad \xi'' = g'(g(x))g'(x)\xi + \mathcal{E}[g'(x)\psi(g(x),\beta) + g'(g(x))\psi(x,\beta)]\xi, \tag{3}$$

and we make the substitution $x \to x/a$ in (3), where $a = -2.5029$ is the Feigenbaum constant. Using $ag(g(\frac{x}{a})) = g(x)$ and $g'(g(\frac{x}{a}))g'(\frac{x}{a}) = g'(x)$ (Ref. 1), we again find Eqs. (1) and (2), but with the coupling function $\psi_1 = g'(\frac{x}{a})\psi(g(\frac{x}{a}),\beta) + g'(g(\frac{x}{a}))\psi(\frac{x}{a},\beta)$. As a result of an N-fold application of this procedure we find

$$x^{(2^N)} = g(x),\quad \xi^{(2^N)} = g'(x)\xi + \mathcal{E}\psi_N(x,\beta)\xi, \tag{4}$$

where the quantities with the superscript (2^N) refer to the time n + 2^N, and the function $\psi_N(x,\beta)$ satisfies the recurrence relation

$$\psi_{N+1}(x,\beta) = g'(\tfrac{x}{a})\psi_N(g(\tfrac{x}{a}),\beta) + g'(g(\tfrac{x}{a}))\psi_N(\tfrac{x}{a},\beta) = \hat{L}\psi_N. \tag{5}$$

Since L is a linear operator, we can seek a solution of (5) in the form

$$\psi_N(x,\beta) = \sum_S C_S(\beta)\nu_S^N\Phi_S(x), \tag{6}$$

where $C_S(\beta)$ are coefficients which depend on the particular initial function $\psi(x, \beta)$, and Φ_S and ν_S are the eigenfunctions and eigenvalues of the operator $\hat{L}$. We interpret each term of series (6) as a definite type of coupling, and we speak in terms of coupling of type A, type B, type C, etc., in order of decreasing modulus of the eigenvalues. The first four functions Φ_S are plotted in Fig. 1; the corresponding eigenvalues are $\nu_A = a = -2.503$, $\nu_B = 2$, $\nu_C = 1/a = -0.400$, and $\nu_D = -0.218$.

Upon a doubling transformation, the weight with which each type of coupling is included in the coupling function $\psi_N(x, \beta)$ is multiplied by the corresponding eigenvalue. At large values of N the couplings of type A and B become predominant, because of the condition $|\nu_{A,B}| > 1$ (provided, of course, that $C_{A,B} \neq 0$). The general question of the possible types of critical behavior thus reduces to studying chains with coupling functions of the form $\psi = C_A\Phi_A + C_B\Phi_B$; i.e., the problem is greatly simplified in comparison with the problem as originally formulated, which contained the arbitrary function $\psi(x, \beta)$. It follows that in

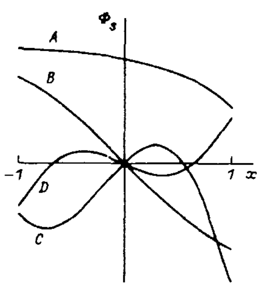

FIG.1

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order to study the critical behavior we must be able to find C_A and C_B for the specific systems. We turn now to one method for determining these coefficients.

3. Mapping (1) has cycles of periods 2, 4, 8, 16, ... We define the multiplier for the 2^N cycle, $\mu_N(\beta)$, as the quantity which shows by how many times a perturbation of the form $e^{i\beta m}$ changes over a period of the cycle against the background of the spatially uniform solution corresponding to this cycle. It can be seen from (4) that the renormalized value of x corresponding to the 2^N cycle is $x_* = 0.5439$, the root of the equation[1] $g(x) = x$. Substituting $x = x_*$ into the equation for ξ, using (6), and assuming N large, we find

$$\mu_N(\beta) = \mu_* + c_A a^N + c_B 2^N, \tag{7}$$

where $\mu_* = g'(x_*) = -1.6012$, and $C_{A,B} = C_{A,B}\Phi_{A,B}(x_*)\mathcal{E}$ If we know the multipliers even for just two different cycles, we can find c_A and c_B from (7).

4. If each cell is coupled in a symmetric fashion with its neighbors on the left and right, then the coupling function at small values of β is of the form $\psi(x,\beta) \approx \psi^0(x)\beta^2$. How does the presence of one type of coupling or another affect the dynamics of the perturbation here? Let us examine some particular cases.

A. $c_A \neq 0$, $c_B = 0$. Since $a < 0$, it follows from (7) that for 2^N cycles with even values of N for $c_A < 0$ and odd values of N for $c_A > 0$, the growing perturbations are those with small values of β. The corresponding interval of β values falls off with increasing N in proportion to $|a|^{-N/2}$. For cycles with odd values of N for $c_A < 0$ and even values of N for $c_A > 0$, the perturbations grow more rapidly as their wave number increases.

B. $c_A = 0$. If $c_B < 0$, then for any 2^N cycle the growing perturbations are those with small values of β, from an interval whose width is proportional to $2^{-N/2}$. If $c_B > 0$, all the 2^N cycles are unstable with respect to perturbations with large wave numbers.

In the case $|c_A| \ll |c_B|$ behavior of type B is found for cycles whose period is not too long; as N increases, this behavior gives way to type A behavior (since $|\nu_A| > |\nu_B|$

5. It has been shown elsewhere that Eq. (1) gives a satisfactory description of all the details of the critical behavior for a broad range of dynamical systems with both discrete and continuous time scales.[1,2] It may thus be expected that the results derived above will apply to an equally broad range of one-dimensional chains. As a specific example we consider a chain of parametrically excited nonlinear oscillators,[3] a case of independent interest in solid state physics, nonlinear optics, and electronics:

$$\ddot{x}_m + k\dot{x}_m + \left(1 + q\cos\frac{2\pi t}{T}\right)x_m + x_m^3 = \mathcal{E}\left(x_{m-1} - 2x_m + x_{m+1}\right). \tag{8}$$

Here x_m is the generalized coordinate of the m-th oscillator; k, q, and T are parameters; and $\mathcal{E}$ is the coupling coefficient between oscillators. At fixed values of q and T, the specially uniform solution of Eq. (8) undergoes a bifurcation (doubling) with decreasing value of the parameter k, exhibiting regimes (cycles) of periods 2T, 4T, 8T, ... (Ref. 3). For q = 4, $2\pi/T = 2.04$, the limit of the sequence of bifurcation values of k is $k_c = 0.4105$. At $k = k_c$ there are cycles with all possible periods $2^N T$. The multipliers of these cycles were found by numerical calculation in the present study for perturbations of the type $e^{i\beta m}$ with $\beta^2 \ll 1$: $\mu_N(\beta) = \mu_* + \mathcal{E}\varkappa_N\beta^2$. The values of $\varkappa_N$ for N = 1, 2, 3, and 4 are 5.12, 4.49, 27.4 and −36.6, respectively. This sequence is reproduced accurately by the expression $\varkappa_N = c_1 a^N + c_2 2^N$ ($c_1 = -1.30$, $c_2 = 0.90$), confirming the arguments in Sections 1-4.

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