RENORMALIZATION GROUP, UNIVERSALITY, AND SCALING IN THE DYNAMICS OF ONE-DIMENSIONAL AUTOWAVE MEDIA

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The behavior of a certain class of one-dimensional autowave media at the threshold of dynamic chaos is studied by the method of the renormalization group. The properties of universality and scale invariance of the parameter space are established, and different types of bifurcations and characteristic regimes of the dynamics of the media of interest are studied.

The concepts of scaling and the renormalization group (RG), formulated in the quantum theory of fields and in the theory of phase transitions, have in recent years been used to study the behavior of nonlinear vibrational systems near the threshold for the appearance of dynamic chaos [1-4]. One speaks about scaling in cases when for some values of the parameters of a system a hierarchy of formations, similar to one another and characterized by different spatial and (or) temporal scales, is realized in the system. The point in parameter space at which this occurs is called a critical point. Obviously, scaling indicates that the dynamic equations of the system are invariant under some transformation, including a change in the scale of the spatial coordinates, time, and dynamic variables. This comprises the RG transformation. The subject of RG analysis involves, first, finding the dynamic equations which are invariant under the RG transformations and, second, studying the effect of perturbations of the parameters, which drive the system out of the critical point, on the form of these equations.

If the form of the dynamic equations at the critical point is determined uniquely by the requirement of invariance under the RG transformation, then the scaling laws are universal: they are determined by the properties of the RG and not by the specific form of the starting equations, describing the dynamics in the region of small spatial and temporal scales.

In this paper the behavior of a class of distributed systems near the threshold for the appearance of dynamic chaos is studied with the help of the RG method. This study is a development of the approach outlined in [5-7].

Consider a distributed medium each point of which is a nonlinear dissipative system (self-excited or excited by an external action) and at each point of which there is a local coupling between the spatially distributed elements. Wave processes whose characteristics are to some extent independent of the initial and boundary conditions and which, following R. V. Khokhlov, are called autowaves, are possible in such media.

The main assumption, which further distinguishes this class of media, is that the individual constituent elements of the medium are systems exhibiting a Feigenbaum sequence of period doublings as a function of some parameter [1, 2]. Since this is one of the typical scenarios of the appearance of chaos in simple systems, autowave media of this type are also very common. They include, for example, 1) an array of externally driven nonlinear dissipative oscillators, which is a model of a crystal in the field of an optical or acoustic wave; 2) a distributed Josephson junction interacting with microwaves; and 3) radioelectronic oscillators with delayed feedback and a number of other systems [7].

1. GENERALIZED MODEL OF THE MEDIUM

As the starting element for constructing a distributed medium we shall choose the simplest system exhibiting a transition to chaos through period doubling, described by the recurrence equation

$$u_{n+1}(x) = f(u_n)\,. \tag{1}$$

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Here u_n is a variable characterizing the state of the medium at the time n, and f(u) is a smooth function, depending on a parameter λ, which maps a segment of the u axis into itself and has on this segment one simple extremum, for example, $f(u) = \lambda - u^2$ for $0 \le \lambda \le 2$. (For convenience we shall always measure the quantity u from the point of the extremum.)

Consider now a one-dimensional medium whose state at the n-th moment of discrete time is described by a function of the spatial coordinate $u_n(x)$. Let the change in the system over one time step be determined by the equation

$$u_{n+1}(x) = F[u_n], \tag{2}$$

where F is a nonlinear operator, which is invariant under spatial translation. In order to be able to regard the medium as a collection of elements described by Eq. (1), we shall require that when operating on an x-independent state u the operator F reduces to the function f(u), satisfying the previously formulated conditions. The assumptions which specify the nature of the coupling between the spatially separated elements of the medium will be indicated below.

According to [1, 8], Eq. (1) can be used to give a quantitative description of the transition to chaos in any typical nonlinear dissipative systems, exhibiting period doubling, including systems described by differential equations; it is only necessary to give an appropriate interpretation of the variable u and the discrete time n.* One would expect that in the same sense the model (2) is applicable to a wide class of distributed media, including systems described by nonlinear partial differential equations.

2. RENORMALIZATION GROUP ANALYSIS

2.1. Construction of the RG Transformation. By analogy with [1] we transform from the operator (2) to an operator describing the evolution of the medium over two time steps and we carry out in addition a scale transformation S:

$$Su(x) = au(bx), \tag{3}$$

where a and b are constants, which are to be determined. As a result we obtain the operator $F_1[u] = SFFS^{-1}[u]$. Repeating this procedure many times, we arrive at the recurrence relation [7]

$$F_n[u] = SF_{n-1}F_{n-1}S^{-1}[u]. \tag{4}$$

This is the RG equation. The operator F_n is the operator describing the evolution of the medium over 2^n time steps (with rescaled scales of u and x). Next, we use Eq. (4) to study the asymptotic behavior of the operator F_n for large n. In this manner we shall obtain information about the evolution of the system over long times, which is the main subject of study in the problem of the transition to chaos.

2.2. Stationary Point of the RG Transformation. We shall examine the case when the "seed" evolution operator has the following form after one step:

$$F_0[u] = \hat{l} f(u), \tag{5}$$

where $\hat{l}$ is a linear integral operator: $\hat{l}u(x) = \int l(x-\xi)u(\xi)d\xi$. We require that the spectrum of this operator $l(k) = \int l(x)e^{-ikx}dx$ be an analytic function of the wave number k and for all real $k \neq 0$ it must satisfy the inequality $|\ell(k)| < 1$. For small k

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*The procedure for transforming from differential equations to the mapping (1) consists of the following. First, the phase space is sectioned by some surface, and the analysis is confined to a discrete set of states of the system at times when the phase trajectory intersects the surface. Because of the dissipative nature of the system any element of the phase volume is strongly compressed over a long time period. For this reason, when describing the dynamics over long time intervals, the mapping of the sectioning plane into itself reduces to an equation of the type (1), and in addition the variable u corresponds to the direction in phase space along which the degree of compression is minimum. (In the region of stochastic regimes one must talk about stretching along the direction u.) If the function f(u) obtained as a result of this procedure satisfies the conditions formulated, then a Feigenbaum scenario of transition to chaos exists in the system.

$$l(k)=1-D^2k^2+O(k^4)\,, \tag{6}$$

where D is a constant. The choice of an evolution operator in the form (5) indicates that a coupling is introduced between the spatially separated elements of the medium which strives to smooth their instantaneous states, i.e., it damps out short wavelength disturbances. It is natural to call a coupling of this type diffusive. The quantity D determines the scale length over which the effect of the coupling propagates over one time step, and can be called the diffusion length.

Let the parameter λ equal the Feigenbaum critical value λ_c; the scale constants a and b are set equal to $-2.5029\ldots$ and $\sqrt{2}$, respectively (λ_c is the accumulation point of period doubling bifurcations and a is one of the universal Feigenbaum constants [1, 2]). Then the sequence of operators F_n, generated by Eq. (4) with the initial condition (5), has a regular limit $G = \lim F_n$. This assertion is proved in [7] for the particular case when the operators F_n act on slightly nonuniform states

$$u(x) = u+\varepsilon v(x)\,, \tag{7}$$

where u is independent of x, and ε is an infinitesimal. We assume, however, that it also holds for arbitrary nonuniform states. As will be evident from the numerical results (Sec. 3), this assumption leads to the correct represenation of the scaling properties of the media under study.

The limiting operator G satisfies the operator equation

$$G[u] = SGGS^{-1}[u] \tag{8}$$

and, according to [7], is approximately expressed in the form

$$G[u] = \exp(D^2\partial^2/\partial x^2)\,g(u)\,, \tag{9}$$

where g is Feigenbaum's function [1]. The operator G is the stationary point of the RG equation in operator space. It is universal in the sense that it is independent of the specific choice of the function f and the operator $\hat{l}$ in the seed operator (5).

2.3. Solution of the RG Equation near the Stationary Point. Consider a medium with diffusive coupling between its constituent elements with a critical value of the parameter λ_c. Assume, further, that we have slightly disturbed it by introducing an additional weak coupling of a different type or by changing λ. (The disturbances do not destroy the translational invariance of the problem!) How will the form of the evolution operator change over a large number of steps?

We shall seek the solution of Eq. (4), close to G, in the form

$$F_n[u] = G[u]+\mu H_n[u],\quad \mu\ll 1\,. \tag{10}$$

Substituting (10) into (4) and neglecting terms of order μ^2, we obtain

$$H_n[u]=SG'(GS^{-1}u)H_{n-1}S^{-1}[u]+SH_{n-1}GS^{-1}[u]\,, \tag{11}$$

where $G'(GS^{-1}u)$ is the Fréchet derivative of the operator G. Equation (11) has the structure $H_n[u]=\hat{M}H_{n-1}[u]$, where $\hat{M}$ is a linear operator. The asymptotic behavior of its solution for large n will be determined by the superposition of the characteristic vectors of the $\hat{M}$ which correspond to the characteristic numbers whose modulus is greater than unity and which we shall call significant.* Thus we arrive at the operator problem for the characteristic vectors and characteristic values

$$\nu H[u]=SG'(GS^{-1}u)HS^{-1}[u]+SHGS^{-1}[u]\,. \tag{12}$$

As shown in Appendix 1, the operator $\hat{M}$ has four significant characteristic vectors $h_i\,[u]$, $i = 0, 1, 2, 3$, which correspond to the characteristic numbers $\nu_0 = \delta = 4.6692\ldots$, $\nu_1 = -1.7698\ldots$, $\nu_2 = 1.4142\ldots$, $\nu_3 = -1.2512\ldots$. For this reason, for the evolution operator over 2^n steps we obtain the expression

$$F_n[u]=G[u]+\Lambda\delta^n\, h_0[u]+\alpha\nu_1^n\, h_1[u]+\beta\nu_2^n\, h_2[u]+\gamma\nu_3^n\, h_3[u], \tag{13}$$

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*Like in Feigenbaum's theory [1, 2], among the set of significant characteristic vectors we must seek those vectors which are associated with infinitesimal substitutions of variables. This will be done below without any additional stipulations.

and in addition the form of the initial disturbance $\varepsilon H_0 = F - G$ is determined only by the values of the constants Λ, α, β, γ. These constants play the role of the significant parameters of the problem. The quantity Λ characterizes the deviation of the controlling parameter λ from the critical point: $\Lambda = \lambda - \lambda_c$, while the parameters α, β, γ are different types of couplings between the elements of the medium, which can exist in addition to the diffusive coupling (see the Appendix). The following important conclusions follow from the relation (13).

Universality. If the starting evolution operator F is close to the stationary point G, then the evolution operator is completely determined over a large number of time steps by the four constants Λ, α, β, γ. For this reason, the regions of different regimes of the dynamics of the medium in the space of these parameters have a universal structure which is independent of the specific form of the starting operator.

Scale Invariance. As is evident from (13), the form of the evolution operator does not change under the substitutions $n \to n + k$, $\Lambda \to \Lambda\delta^{-k}$, $\alpha \to \alpha\nu_1^{-k}$, $\beta \to \beta\nu_2^{-k}$, $\gamma \to \gamma\nu_3^{-k}$, where k is any integer. Therefore the same regimes of the dynamics of the medium must be realized at the point $\Lambda\delta^{-k}$, $\alpha\nu_1^{-k}$, $\beta\nu_2^{-k}$, $\gamma\nu_3^{-k}$ as at the point Λ, α, β, γ, but with the spatial scale increased by a factor of $2^{k/2}$ and the time scale increased by a factor of 2^k.

3. INVESTIGATION OF THE STRUCTURE OF PARAMETER SPACE: LATTICE MODELS AND NUMERICAL EXPERIMENTS

By virtue of the universality established in the preceding section, in order to reveal the structure of the parameter space near the critical point (λ_c, 0, 0, 0) any specific system of the class under study can be studied. For numerical experiments the lattice models, in which the time and spatial coordinates are discretized, are most convenient.

A model of a medium with a purely diffusive coupling between its constituent elements ($\alpha = \beta = \gamma = 0$) can be obtained by specifying on a discrete lattice an evolution operator of the form (5). A good example is a model described by the equation

$$u_{n+1,m} = \hat{l}(\lambda - u^2_{n,m}), \tag{14}$$

where m is the discrete spatial coordinate, n is the discrete time, and $\hat{l}$ is an operator expressing the averaging over three lattice sites:

$$\hat{l}v_m = (1/3)(v_{m-1} + v_m + v_{m+1}) . \tag{15}$$

The dynamics of models of this type were studied in [5, 7]. Here we shall study more complicated models of media in which each of the types of coupling characterized by the parameters α, β, γ is introduced in turn. The general principle for constructing such models consists of adding to Eq. (14) additional terms which vanish in the case of spatially uniform states. The structure of these terms is chosen based on the results of Appendix 1 corresponding to the properties of the operators $h_l[u]$ (the nature of the dependence on u, symmetry or antisymmetry with respect to change in the orientation of the coordinate axis). We shall confine ourselves to the analysis of systems with an asymptotically long length, chosen so as not to affect the bifurcation values of the remaining parameters.

3.1. Consider a medium in which there exists a type of coupling characterized by the parameter α. In order to introduce a coupling of this type into the model (14), we must add to the expression in the parentheses a nonlocal term which changes sign as the orientation of the coordinate axis changes and which depends on u in accordance with the characteristic function $\Phi_A(u)$ (Appendix 1). Thus we shall study the following model:

$$u_{n+1,m} = \hat{l}[\lambda - u^2_{n,m} + \alpha(1 - 0{,}176u_{n,m})(u_{n,m+1} - u_{n,m-1})] , \tag{16}$$

where $\hat{l}$ is the averaging operator (15). We give the boundary conditions in the form

$$u_{n,0} = c_1, \quad u_{n,L} = c_2 , \tag{17}$$

where L is the length of the system, and c_1 and c_2 are constants ("fixed points").

The results of the numerical study of the model (16) are shown in Figs. 1 and 2. Figure 1 shows part of the plane of the parameters (α, λ), adjacent to the critical point (0,

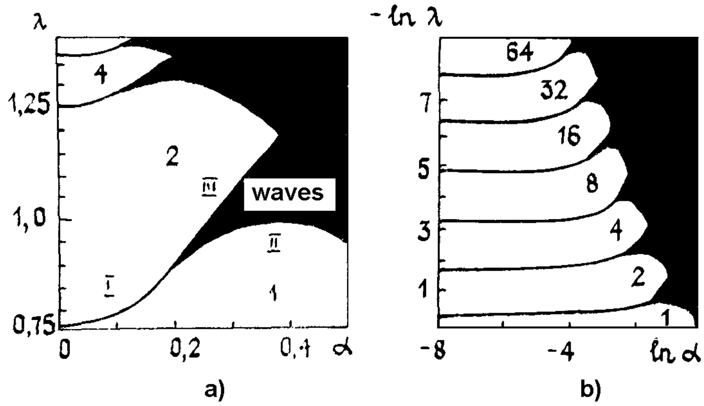

Fig. 1. The parameter plane of the model (16). The light colored regions correspond to regimes which are spatially uniform far from the boundaries of the medium. The dark regions are substantially nonuniform regimes. The same labelling is used in Figs. 4 and 7.

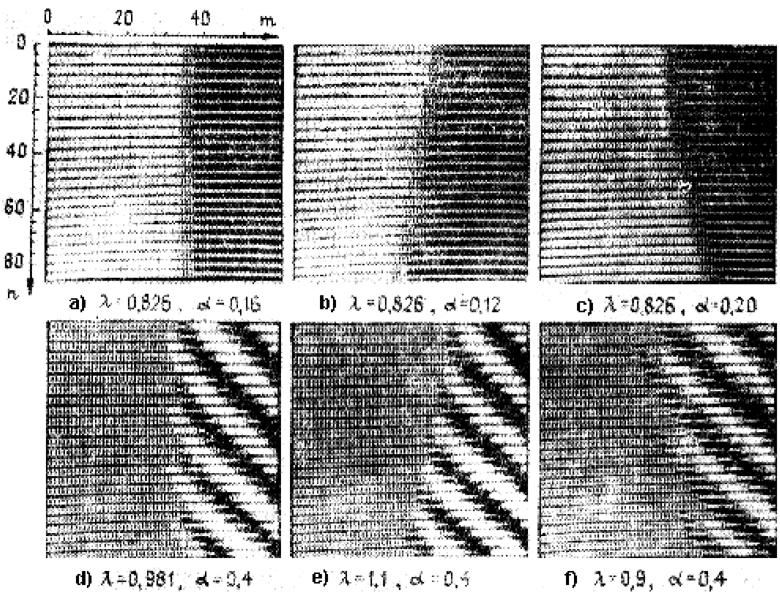

Fig. 2. Space—time diagrams illustrating the dynamics of the medium. The coordinate is plotted along the horizontal axis and time is plotted along the vertical axis downwards. The values of u at different points are characterized by the brightness of the symbols.

λ_c). Figure 2 illustrates the space—time dynamics of the medium in different regions of the parameter plane. Analysis of these results reveals three types of bifurcations.

If we are on the bifurcation line I (Fig. 1), then a situation in which to the left of the narrow transitional region the elements are in a state with the time period 1 while to the right they are in a state with a time period 2 is possible (Fig. 2a). If we move off of the bifurcation line to the left or right, then the transitional region will begin to move from right to left, so that finally the entire medium will be in a state with the period 2 (Fig. 2b). If, on the other hand, we move off this line to the right or downwards, then the transitional region will begin to move to the right and the entire medium will arrive at a state with the time period 1 (Fig. 2c).

If we are on the bifurcation line II, a situation in which the medium contains a region in which the elements are in the state with period 1 (on the left) and a region of traveling waves (on the right) (Fig. 2d) is possible. They are separated by a transitional region which plays the role of an internal source of waves traveling to the right. As λ is increased, the source region begins to move to the left, so that ultimately the entire medium will be entrained into a motion of the traveling-wave type (Fig. 2e). As λ is decreased the source region begins to move to the right, and ultimately the medium is in the state with period 1 (Fig. 2f).

The bifurcation line III separates the traveling-wave regime and the regime with period 2. Let us follow the change in regimes as this line is crossed. First, let the values of α be greater than the bifurcation value. Then, as already mentioned, the medium contains a source region which emits waves to the right, while it itself moves to the left. Having reached the boundary of the medium, the source stops, but does not vanish, so that the traveling-wave regime continues to exist. When the parameter α approaches the bifurcation point, the wavelengths of the waves which are emitted by the source located at the boundary of the medium increases and approaches infinity. After the bifurcation point is crossed, the traveling waves are no longer emitted, and the medium remains in the state with the time period 2.

It is shown in Appendix 2 that the bifurcations I and II are determined by a change of the instability of the spatially uniform state from the convective instability to an absolute instability [9] and, thus, do not have exact analogs in the theory of concentrated systems. In particular, these bifurcations characteristically have an unusual collection of properties of soft and rigid transitions: on the one hand, they are reversible (no hysteresis), while on the other they are accompanined by a finite change in the state of the medium with a small change in the controlling parameters. The bifurcation III is associated with phenomena at the boundary of the medium, where the deviation from the spatial uniformity is significant and therefore is not observed within the framework of the instability analysis performed in the Appendix.

We note that the change in the sign of the parameter α is equivalent to a change in the orientation of the coordinate axes; in this case the form of the bifurcation lines remains unchanged.

According to the results of the RG analysis, the plane of the parameters must have a scale-invariant structure near the critical point $(0, \lambda_c)$. Indeed, it is evident from Fig. 1, that the discussed configuration of regions and bifurcation lines is repeated many times on smaller scales as the critical point is approached. Asymptotically, the form of the regions becomes universal, which is especially clearly seen in logarithmic coordinates (Fig. 1b). The similarity coefficients correspond to the expected ones (δ along the λ axis and ν_1 along the α axis).

Figure 3 demonstrates the scaling of the dynamic regimes of the medium, realized at two points of the parameter plane, for which the values $\lambda - \lambda_c$ differ by a factor of δ^2, while the values of α differ by a factor of ν_1^2. A comparison of Figs. 3a and 3b shows that the structures arising in the medium are similar, and in addition the spatial scale differs by a factor of 2 while the time scale differs by a factor of 4. We note that a transformation of the parameters by a factor of δ^r and ν_1^r with odd r must be supplemented by a change in the orientation of the coordinate axis. (This is liked with the fact that $\nu_1 < 0$ and for an r-fold renormalization α changes sign.)

3.2. We shall now study a medium which contains a coupling of the type characterized by the parameter β. The introduction of this type of coupling corresponds to the situation when aside from the diffusion there exists a drift with a constant velocity. The linear operator appearing in (5) must be replaced in this case by $\hat{l}\hat{\Delta}_\beta$, where $\hat{\Delta}_\beta$ is the translation operator: $\hat{\Delta}_\beta v(x) = v(x-\beta)$. The spectrum of the operator $\hat{l}\hat{\Delta}_\beta$ will have the form $\ell(k)e^{i\beta k} = 1 + i\beta k - (D^2 + \beta^2/2)k^2 + O(k^2)$. In order to construct a lattice model, we shall modify the the operator $\hat{l}$ in Eq. (14) so that its spectrum would correspond for small k to the expression presented. For this, we can set

$$\hat{l}v_m = p_1 v_{m-1} + (1-p_1-p_2)v_m + p_2 v_{m+1}, \tag{18}$$

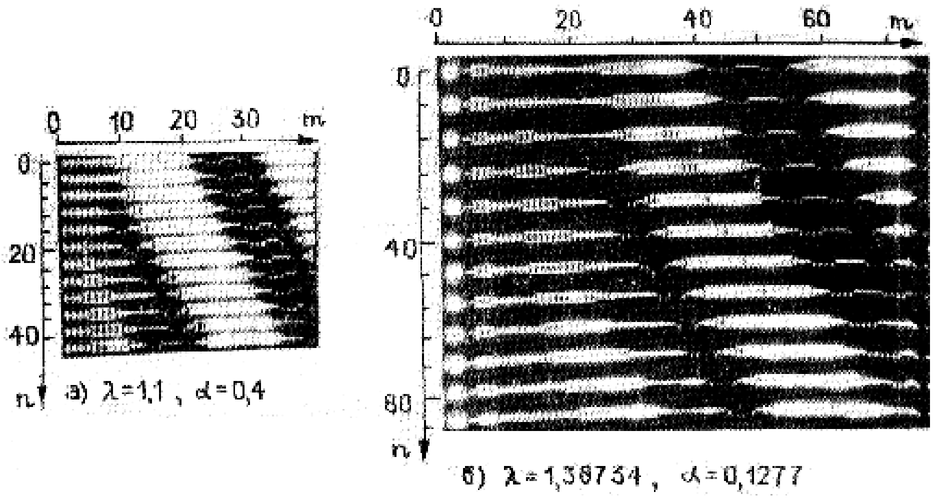

Fig.3

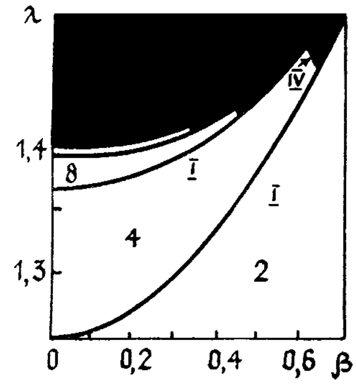

Fig.4

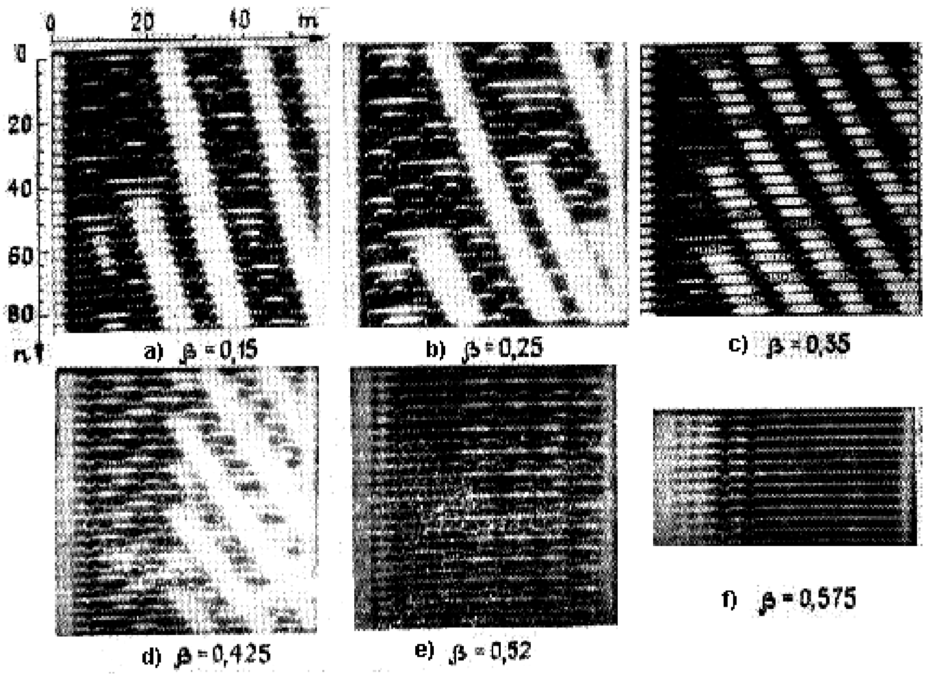

Fig.5

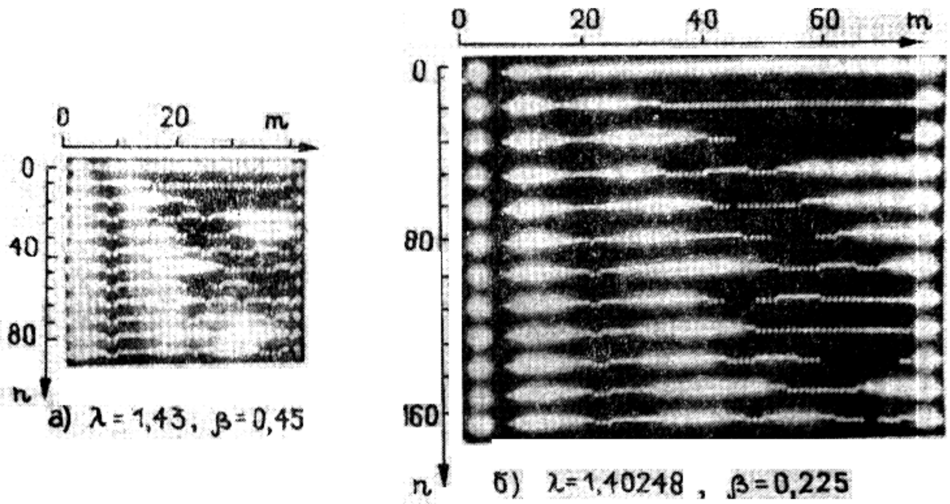

Fig.6

where

$$p_{1,2}=(1/2)\,((1/2)\mp\beta+\beta^2),\qquad D=(1/2).$$

We shall give the boundary conditions at the ends of the medium in the form (17), as before.

Figure 4 shows the structure of the parameter plane (β, λ) near the cirtical point (0, λ_c). Like in the preceding case, the number 1 denotes bifurcation lines corresponding to a change in the direction of motion of the front at which switching between states with different time periods occurs. This bifurcation is accompanied by a change in the nature of the instability of the spatially uniform regime from the convective to the absolute instability (Appendix 2). The number IV denotes a bifurcation which consists of the following. If we are on the left of the bifurcation line, then at the boundary point near the left edge of the medium the dependence of u on the coordinate has the form of oscillations which decay away from the edge. As the bifurcation line is approached the spatial damping decrement decreases and vanishes on the line itself. After the bifurcation the entire medium is in a nonuniform state, which is spatially periodic far from the left boundary. The time period of the regime remains equal to that prior to the bifurcation.

Figure 5 gives a general idea of the space—time dynamics of the medium in different regions of the parameter plane (β, λ). All figures (a-f) refer to the same value λ = 1.45 and are arranged in order of increasing parameter β. At β = 0 we have a medium with a symmetric diffusive coupling, in which a stochastic regime is realized [7]. The stochastic dynamics remain also for low, nonzero values of β (chaotic traveling waves in Figs. 5a and 5b). As β increases the chaos vanishes and a quasiperiodic regime of traveling waves (Figs. 5c and d), which are generated at the left edge of the medium and travel to the right, appears. As β is further increased, the traveling waves also vanish and a state of the medium with a time period 4, which is characterized by a complicated spatial structure (Fig. 5e), arises. After the bifurcation line I is crossed, a state which is uniform far from the edge with a time period 2 (Fig. 5f) is realized. We note that the transformation $\beta \to -\beta$ is equivalent to a change in teh direction of the coordinate axis.

It is evident from Fig. 4 that the pattern of regions near the critical point has a scale-invariant structure. It is reproduced when the scale along the coordinate axes λ and β is changed by a factor of δ and ν_2 in accordance with the results of the RG analysis.* Figure 6 illustrates the similarity of the structures realized in the medium at two points of the parameter plane, for which the values of $\lambda - \lambda_c$ differ by a factor of δ^2 and the value of β differs by a factor of ν_2.

3.3. The last case which remains to be discussed is a medium with a coupling determined by the parameter γ. In order to introduce this type of coupling into the model (14) a symmetric nonlocal term, whose dependence on u corresponds to the characteristic function $\Phi_A(u)$, must be added to the expression in parentheses (Appendix 1). We shall study the model

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*We call attention to the difference from the preceding case (Sec. 3.1): since $\nu_2 > 0$, under a similarity transformation the orientation of the spatial coordinates does not change.

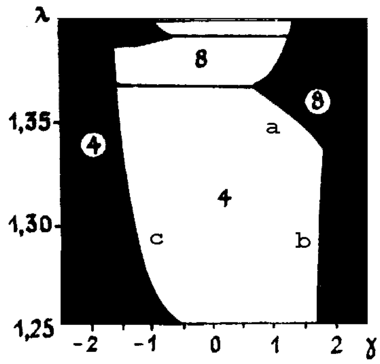

Fig.7

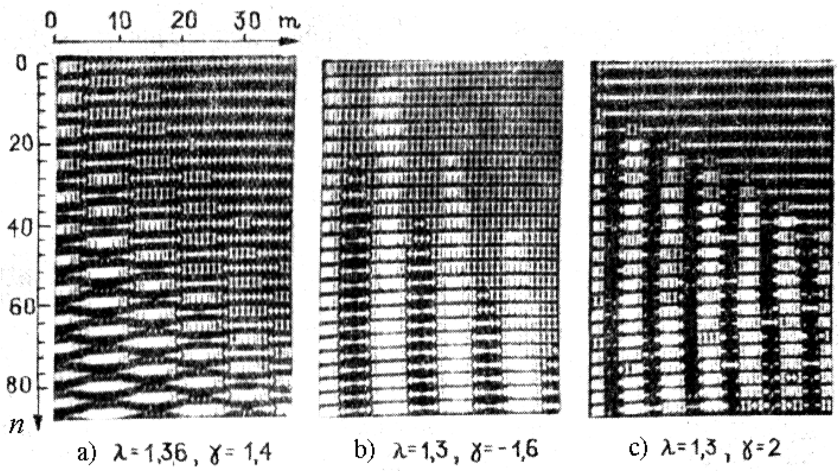

Fig.8

$$u_{n+1,m}=\hat{l}[\lambda-u_{n,m}^2+\gamma(1-0{,}176u_{n,m})(u_{n,m+1}-2u_{n,m}+u_{n,m-1})], \tag{19}$$

where

$$\hat{l}=\hat{l}_1^2,\quad \hat{l}_1 v_m=(1/3)(v_{m-1}+v_m+v_{m+1}).$$

Unlike the preceding cases, we shall give the boundary conditions in the form $u_{n,0}=u_{n,1}$, $u_{n,L+1}=u_{n,L}$ (free ends).*

Figure 7 shows the part of the parameter plane (γ, λ) which is adjacent to the critical point (0, λ_c). The large light-colored region at the center of the figure is the region of the spatially uniform regime with the time period 4. We shall clarify the nature of the bifurcations observed as we leave this region through different boundaries.

The transition through the top boundary corresponds to a time-period-doubling bifurcation of the spatially uniform solution, while the transiton through the bottom boundary corresponds to a decrease of the period by a factor of 2.

In the case of a transition through the lateral boundaries of the region, bifurcation of a loss of symmetry with the appearance of spatially nonuniform regimes occurs. There are three different variants of these bifurcations (lines a, b, and c in Fig. 7). The first bifurcation (a) is accompanied by the appearance of a periodic standing wave, whose time period is two times longer than in the starting regime (Fig. 8a). This bifurcation is soft, and there is no hysteresis. The two other lines (b, c) correspond to rigid bifurcations of symmetry loss. As a result, shorter wavelength, than in the case (a), spatial structures (Figs.

*For the system (19) with a symmetric coupling results analogous to those described below are also obtained in the case of fixed ends. In this case, however, the length of the system must be increased in order to reveal the asymptotic behavior as a function of L.

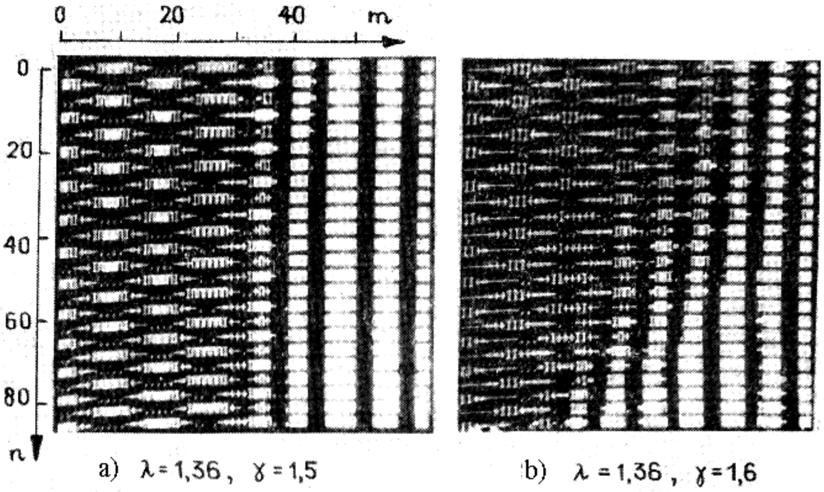

Fig.9

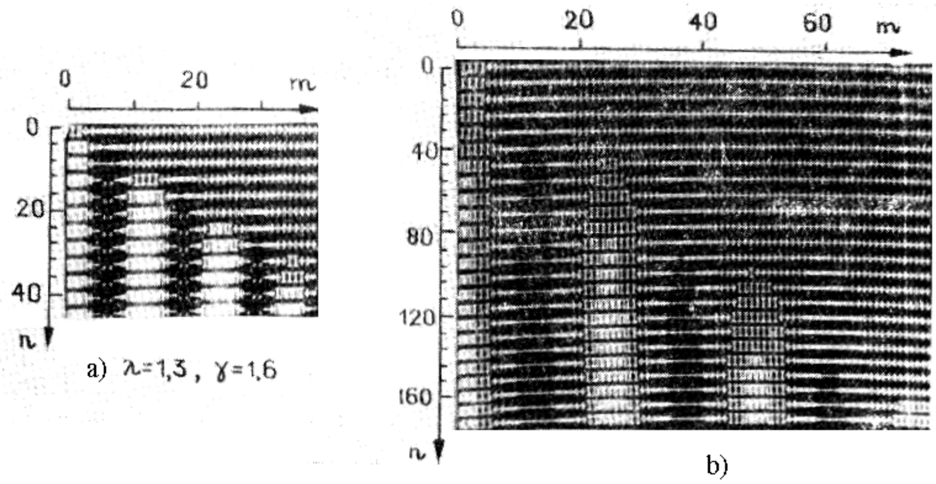

Fig.10

8b and c) appear in a jumplike manner. The spatially nonuniform regime that appears has in the case (b) the same temporal period as in the starting regime, while in the case c, apparently, the regime immediately becomes stochastic.

Near the point of intersection of the bifurcation lines (a) and (c) regimes in which th short wavelength perturbations compete with the long wavelength perturbations can be realize In some region of finite width in the (γ, λ) plane both types of perturbations coexist, bein spatially separated (Fig. 9a). Outside this region one of them displaces the other (Fig. 9b

In accordance with the expected property of scale invariance, the structure of the para eter plane (γ, λ) transforms into itself under a change of scale by a factor of δ along the λ axis and by a factor of ν_3 along the γ axis. Since $\nu_3 < 0$, the change in scales is accompanied by a mirror reflection of the pattern of the regions relative to the vertical axis Figure 10 demonstrates the scaling structures, realized in the medium at similar points of the plane (γ, λ), in which the quantities $\lambda - \lambda_c$ differ by a factor of δ^2, while γ differs by a factor of ν_3^2.

4. CONCLUSIONS

The method developed above is applicable to distributed media which satisfy two conditions: 1) in the spatially uniform case the medium exhibits a Feigenbaum sequence of time period doublings and 2) the main (but possibly not the only) factor providing coupling between the elements of the medium is diffusion, which smoothes the short-wavelength disturbances.

The role of the theory constructed above with respect to the given class of media is similar to the role of Feigenbaum's theory for finite-dimensional systems. The development of the RG method, however, makes it more profound, leading, for example, to the idea of scaling of spatial structures along the path to chaos. On the methodological level, this makes the concept of the theory of dynamic systems similar to the concepts of the theory of phase transitions and quantum field theory, which facilitates the formulation of the general physical concepts of the renormalization group and scaling.

The presence of a large number (four) of free parameters in the theory may appear to be somewhat discouraging. Actually, this is a consequence of the generality of the approach, in the analysis of specific problems not all parameters may be significant. For example, if the coupling between elements of the medium is symmetric with respect to a change of the orientation of the spatial coordinate, then $\alpha = \beta = 0$ and only two free parameters γ and λ remain.

Possible directions of further investigations include systematic substantiation of the approach (rigorous analysis of the operator equations (8) and (12)), as well as experimental investigations of distributed systems at the threshold for the appearance of chaos from the viewpoint of the laws of the critical behavior found here. It would be interesting to extend the renormalization group appraoch to two- and three-dimensional problems, and also to study media consisting of elements exhibiting different types of critical behavior [3, 4].

APPENDIX

As already mentioned, in the media studied here it is always assumed that there exists a diffusive coupling, giving rise to decay of the short wavelength perturbations. For this reason, in order to understand the characteristic features of the evolution of these media it is useful to study some questions regarding the dynamics of nearly spatially uniform states. In part I of the Appendix, the operator equations obtained in the main text are studied in application to this class of states. The results of this analysis are sufficient to formulate uniquely the hypotheses of universality and similarity (A.2) and to construct models which can be employed to check these hypotheses numerically (A.3). In part II of the Appendix the instability of spatially uniform states is studied. This supplements the results of the numerical experiments presented: the nature of the observed bifurcations is clarified and a method for finding bifurcation lines is indicated.

A.1. Analysis of the Operator Equations for the Class of Weakly Nonuniform States. We introduce the following notation for weakly nonuniform states of the type (7):

$$u(x) = \begin{bmatrix} u \\ v_k e^{ikx} \end{bmatrix} = u + \varepsilon \int v_k e^{ikx} dk, \tag{A.1}$$

where the Fourier representation is used for the variable part v(x).

As an example, consider the effect of the operator $G[u] = \exp(D^2\partial^2/\partial x^2)g(u)$ on this state Employing the notation introduced we have

$$G\begin{bmatrix} u \\ v_k e^{ikx} \end{bmatrix} = \begin{bmatrix} g(u) \\ v_k g'(u) e^{-D^2k^2+ikx} \end{bmatrix}. \tag{A.2}$$

Let us verify that the operator (A.2) satisfies Eq. (8). Indeed, operating on any weakly nonuniform state with the operator on the right side of (8), we obtain the same result as with the operator G: $SGGS^{-1}\begin{bmatrix} u \\ v_k e^{ikx} \end{bmatrix} = SGG\begin{bmatrix} u/a \\ a\; v_k \exp(ikx/\sqrt{2}) \end{bmatrix} = SG\begin{bmatrix} g(U/a) \\ a^{-1} v_k g'(u/a) \exp(-D^2k^2/2 + ikx/\sqrt{2}) \end{bmatrix} = S\begin{bmatrix} g(g(u/a)) \\ a^{-1} v_k g'(u/a) g'(g(u/a)) e^{-D^2k^2/2+ikx/\sqrt{2}} \end{bmatrix} = \begin{bmatrix} g(u) \\ v_k g'(u) e^{-D^2k^2+ikx} \end{bmatrix}$. Here we took into account the fact that by definition of the function g(u), ag(g(u/a)) = g(u) and g'(u/a)g'(g(u/a)) = g'(u).

We now consider Eq. (12). First we note that the result of the action of the operator H on such a state will have the form

$$H[u(x)] = \begin{bmatrix} h(u) \\ \varphi(k, u) v_k e^{ikx} \end{bmatrix}, \tag{A.3}$$

where h(u) and $\varphi(k, u)$ are functions which are determined by the structure of the operator H[u]. Without loss of generality, we impose on the function $\varphi(k, u)$ the condition $\varphi(0, u) = 0$.* Next, we calculate the Fréchet derivative in (12) by employing its definition [10]:

$$G[u(x)+\tilde{u}(x)]=G[u]+G'[u]\tilde{u}+O(\tilde{u}) . \tag{A.4}$$

As a result we obtain

$$G'[u]=\begin{bmatrix} g'(u) & 0 \\ 0 & e^{-D^2k^2+ibx}g'(u) \end{bmatrix}. \tag{A.5}$$

Taking into account (A.3) and (A.5), from (12) we obtain

$$\nu\begin{bmatrix} h(u) \\ \varphi(k,u)e^{ikx} \end{bmatrix}=\begin{bmatrix} ag'(g(u/a))\,h(u/a)+ah(g(u/a)) \\ e^{-D^2k^2/2+ikx}\left(\varphi\left(\frac{k}{\sqrt{2}},\frac{u}{a}\right)g'\left(g\left(\frac{u}{a}\right)\right)+\varphi\left(\frac{k}{\sqrt{2}},g\left(\frac{u}{a}\right)\right)g'\left(\frac{u}{a}\right)\right) \end{bmatrix}. \tag{A.6}$$

It is evident from here that there exist two classes of characteristic vectors: 1) $h \neq 0$, $\varphi = 0$, and 2) $h = 0$, $\varphi \neq 0$. The characteristic vectors of the first class are independent of the wave number k; they correspond to those disturbances of the stationary point (the operator G) which affect the individual elements of the medium, but not the nature of the coupling between them. Conversely, disturbances of the second class modify the nature of the coupling between the elements, which is indicated by the presence of the dependence on k.

Examining vectors of the first class, we obtain for the function h the same equation as in Feigenbaum's theory [1] (the top line in (A.6)). It gives a unique real characteristic function h(u) with the characteristic vector $\nu_0 = \delta = 4.6692\ldots$ and

$$h_0[u]=\begin{vmatrix} h(u) \\ 0 \end{vmatrix}.$$

In the case of vectors of the second class, we shall seek the solution of the equation for the function $\varphi(k, u)$ (bottom line in (A.6)) in the form $\Psi(k)\Phi(u)$. Then the variables will separate and we can write

$$\begin{gathered} \eta\Phi(u)=g'(g(u/a))\Phi(u/a)+g'(u/a)\Phi(g(u/a)), \\ \xi\Psi(k)=e^{-D^2k^2/2}\,\Psi(k/\sqrt{2}). \end{gathered} \tag{A.7}$$

The first equation has real characteristic functions $\Phi_A(u)$ and $\Phi_B(u)$, which behave near the origin of the coordinates as $1 + O(u)$ and $u + O(u^3)$, respectively [6]. They correspond to the characteristic numbers $\eta_A = a$ and $\eta_B = 2$ (this is easy to check, by examining the equation in the region of small u). To these two characteristic functions there are associated two different methods for introducing the coupling between the elements of the medium, which we shall denote by A and B [6] (see [11] for a qualitative interpretation of these methods).

The characteristic functions of the second equation (A.7) have the form $\Psi_m(k) = (ik)^m \times e^{-D^2k^2}$, m = 1, 2, 3, ... Here we took into account the imposed condition $\varphi(0, u) = 0$, and also the fact that $\Psi(k) = \Psi^*(-k)$, since $\varphi(k, u)$ is the Fourier transform of a real function. Substituting $\Psi_m(k)$ into the equation it is evident that the characteristic numbers are $\xi_m = 2^{-m/2}$. The characteristic functions with even m correspond to a coupling which is symmetric with respect to a reversal of the orientation of the coordinate axis, while odd m correspond to antisymmetric coupling.

Combining the results of the analysis of the two equations (A.7) we find that the real characteristic vectors of the second class $h_s[u]=\begin{bmatrix} 0 \\ \varphi_s e^{ikx} \end{bmatrix}$ are characerized by the following characteristic numbers and characteristic functions $\varphi(k, x)$.†

*If $\varphi(0, u) \neq 0$, then the functions h and φ can be redefined as follows: $h'(u)=h(u)-\varepsilon\varphi(0,u)$, $\varphi'(k,u)=\varphi(k,u)-\varphi(0,u)$.

†The diffusive coupling is not included in this list, since it plays in our analysis a special distinguished role, determining the characteristic spatial scale of the evolution operator – the diffusion length. In accordance with the classification introduced, the diffusion coupling is a symmetric coupling of type B; it corresponds to the characteristic number $\nu_4 = 1$ and characteristic function $\varphi_4=k^2e^{-D^2k^2}\Phi_B(u)$. The perturbation of the evolution operator determined by this characteristic function is obviously equivalent to the introduction of a correction to the diffusion length.

1) Antisymmetric coupling of the type A: $\nu_1=\eta_A\xi_1=(a/\sqrt{2})$, $\varphi_1 = ike^{-D^2k^2}\Phi_A(u)$.

2) Antisymmetric coupling of the type B: $\nu_2=\eta_B\xi_1=\sqrt{2}$, $\varphi_2= ike^{-D^2k^2}\Phi_B(u)$.

3) Symmetric coupling of the type A: $\nu_3=\eta_A\xi_2=(a/2)$, $\varphi_3=-k^2e^{-D^2k^2}\Phi_A(u)$.

A.2. Nature of the Instability of Spatially Uniform Regimes. Assume that for some value λ in an extended medium a uniform solution with the temporal period N can exist:

$$u_{n+1}=f(u_n),\quad u_{n+N}=u_n. \tag{A.8}$$

We shall examine the dynamics of a small localized perturbation of this sŏlution v(x). We shall be interested in the conditons under which there appears an absolute instability, for which the perturbation grows and encompasses an increasingly larger section of the medium, including the region of its initial localization.

To follow the evolution of the perturbation, it is necessary to solve simultaneously (A.8) and the variational equation obtained by linearizing (2). It is convenient to solve the latter in the Fourier representation, setting $v(x)=\int v(k)e^{ikx}dk$. Examining a perturbation of the form $v(k)e^{ikx}$, we find the change in its amplitude over the period $v_{n+N}(k) = \mu v_n(k)$, where

$$\mu=\mu(k). \tag{A.9}$$

The relation (A.9) is a dispersion equation, since it relates the wave number k and the quantity μ, determining the time evolution of the perturbation. We shall employ the method of [12] to find the conditons under which an absolute instability appears. It consists of the fact that first the surfaces in the parameter space which can serve as boundaries of the absolute instability and defined by the equations

$$|\mu(k)|=1,\quad \mu'(k)=0 \tag{A.10}$$

are sought. In so doing, the parameter space is divided into regions in which the nature of the solutions must be determined from additional considerations (for example, from the topology of the contour of integration in the k plane). In our case, it turned out to be more convenient to do this by studying directly the evoltuion of the perturbations in a numerical experiment.

We analyzed the instability of the spatially uniform solutions with the time period $N = 2^r$, r = 0, 1, ..., 6 for the models (16), (18), and (19).

For the model (16) the dispersion equation has the form

$$\mu=\left(\frac{1+2\cos k}{3}\right)^N\prod_{n=1}^{N}[-2u_n+2i\alpha(1-0{,}176u_n)\sin k]. \tag{A.11}$$

For fixed α we found numerically by Newton's method values of k and λ ensuring that the relations (A.10) hold. Two possible variants of the appearance of an absolute instability were found:

$$\mu=-1,\quad \mu'=0,\quad k\text{ — purely imaginary}, \tag{A.12}$$

$$|\mu|=1,\quad \mu'=0,\quad \mu,\ k\text{ — complex}. \tag{A.13}$$

In the first case at the threshold of the absolute instability $\mu = -1$, so that the regime appearing after bifurcation will have a doubled time period. In the second case μ is complex and the regime which appears has the character of traveling waves. The bifurcation lines I shown in Fig. 1 in the (α, λ) plane were calculated with the help of (A.12), while the lines II were calculated with the help of (A.13).

For the model (18) the dispersion equation is as follows:

$$\mu=[1/2-\beta^2+(1/2+\beta^2)\cos k+i\beta\sin k]^N\prod_{n=1}^{N}(-2u_n). \tag{A.14}$$

From here we find numerically the lines in the (β, λ) plane on which the relations (A.12) hold. They are denoted in Fig. 4 by the number 1.*

*The lines in Fig. 4 marked with the numeral IV are of a different nature, but can also be found by analyzing the dispersion equation $\mu = \mu(k)$. The corresponding conditions are obvious from the description of this bifurcation described in the main text: $\mu = 1$, Im k = 0.

The dispersion equation for the model (19) is as follows:

$$\mu=\left(\frac{1+2\cos k}{3}\right)^{2N}\prod_{n=1}^{N}\left[-2u_n+2\gamma(1-0{,}176u_n)(\cos k-1)\right]. \qquad (A.15)$$

Unlike the two preceding cases, it has real coefficients. In this case the conditions (10) are satisfied with real μ and k. The following variants of bifurcations are realized:

$\mu = -1$, $k = 0$ (top horizontal boundary of the region of stability in Fig. 7);

$\mu = 1$, $k = 0$ (bottom horizontal boundary);

$\mu = -1$, $k \neq 0$ (line a in Fig. 7); and,

$\mu = 1$, $k \neq 0$ (lines b and c).

LITERATURE CITED

1. M. Feigenbaum, Usp. Fiz. Nauk, 141, No. 2, 343 (1983).
2. E. B. Vul, Ya. G. Sinai, and K. M. Khanin, Usp. Mat. Nauk, 39, No. 3, 3 (1984).
3. B. Hu and J. Rudnik, Phys. Rev. Lett., 48, No. 24, 1645 (1982).
4. S. Ostlund, D. Rand, J. Sethna, and E. Siggia, Physica D, 8, No. 3, 303 (1983).
5. S. P. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Radiofiz., 25, No. 11, 1364 (1982).
6. S. P. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Fiz., 27, No. 6, 87 (1984).
7. S. P. Kuznetsov and A. S. Pikovskii, Izv. Vyssh. Uchebn. Zaved., Radiofiz., 28, No. 3, 308 (1985).
8. P. Collet, J. P. Eckmann, and H. Koch, J. Stat. Phys., 25, No. 1, 1 (1981).
9. E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics [in Russian], Nauka, Moscow.
10. V. Hatson and J. Pim, Applications of Functional Analysis and the Theory of Operators [Russian translation], Mir, Moscow (1983), p. 136.
11. S. P. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Radiofiz., 28, No. 8, 991 (1985).
12. A. P. Kuznetsov, Pis'ma Zh. Tekh. Fiz., 8, No. 15, 941 (1982).