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Tricriticality in two-dimensional maps

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Tricritical behaviour intrinsic to one-dimensional maps with a quartic extremum near the onset of chaos is found in the modified Hénon map and the Zaslavsky map. The tricriticality appears in two-dimensional maps only as a three-parameter phenomenon while the one-dimensional maps may demonstrate the tricriticality also in the two-parameter case.

It is well known that a wide class of nonlinear systems demonstrate transition to chaos via a period doubling cascade when a control parameter is changed. Among them there are the one-dimensional maps with a quadratic extremum and more complicated nonlinear systems – the Hénon map, Lorenz and Rossler models, the driven nonlinear oscillator and so on. Near the onset of chaos they show the Feigenbaum quantitative universality [1,2].

One may expect that other, more complicated, types of universal behaviour appear in nonlinear systems depending on more than one parameter. For instance, let us consider the smooth one-dimensional map having an extremum and depending on three parameters. In the parameter space the line may exist, where the second and the third derivatives at the extremum point are equal to zero. If the period doubling cascade occurs at this line, the law of its convergence is specific for the map with a quartic, rather than a quadratic, extremum. The convergence rate is 7.2847 instead of the usual Feigenbaum constant 4.6692 (see ref. [3]). The maps having more than one extremum exhibit a similar situation even in the two-parameter case. Really, the condition that a quadratic extremum is mapped to another one is given by one equation. It defines the line on the plane of two parameters, where the twice iterated map $f(f(x))$ has a quartic extremum. So, the period doubling cascade (if it is observed at this line) converges again with the rate intrinsic to the quartic map.

The limit point of the period doubling cascade found under these additional conditions is called *tricritical*. This term was introduced by Chang, Wortis and Wright [4] by analogy to phase transition theory. These authors investigated the one-dimensional two-parameter map

$$x_{n+1}=1-Ax_n^2-Bx_n^4 \tag{1}$$

and found that there are Feigenbaum critical lines (locus of period doubling accumulations) and tricritical points being the ends of these lines. The topography of the parameter plane near all the tricritical points is similar and obeys the two-parameter scaling: the structure is reproduced under parameter rescaling by 7.2847 and 2.8571 along suitable coordinate axes. These results were explained using renormalization group (RG) analysis. It repeats the Feigenbaum approach but deals with another fixed point of the functional RG equation.

Having in mind an analogy with Feigenbaum's theory and the RG results, one could expect that the tricritical behaviour is universal and widespread in nonlinear systems exhibiting transitions to chaos. In fact, the tricriticality was found in several one-dimensional maps (quartic map, cubic map, circle map [4–6], but I do not know of any works where the tricriticality would appear in more complicated systems with a multi-dimensional phase space, even in two-dimensional maps. I think this is not accidental because the question of the tricritical universality is not trivial. It is shown in this Letter that the tricriticality appears typically in two-dimensional maps only as a three-parameter phenomenon in contrast to one-dimensional maps demonstrating also the two-parameter tricriticality. Examples of two-dimensional maps with tricritical behaviour will be considered. Such maps may appear as Poincaré return maps for nonlinear differential equation systems with a three-dimensional phase space. Hence, the tricriticality can be realized in the three-parameter families of these systems too.

Let us recall briefly the RG analysis results for tricriticality. Following Feigenbaum, the RG transformation is defined for the one-dimensional map f:

$$R: f(x) \to a f(f(x/a)) . \qquad (2)$$

Here a is a scaling factor which must be evaluated. Besides the known fixed point found by Feigenbaum ($g(x) = 1 - 1.5276x^2 + 0.1048x^4 + ...$, $a = -2.5029...$), the RG transformation R has the fixed point

$$g_T(x) = 1 - 1.8341x^4 + 0.0130x^8 + 0.3119x^{12} + ... , \qquad (3)$$

responsible for tricritical behaviour. The corresponding scaling constant is $a_T = -1.6903029714$.

The RG operator R, linearized near this fixed point, has three essential eigenvalues in its spectrum which are greater than unity in modulus and are not connected with infinitesimal variable changes:

$$\delta_{T1} = 7.284686217, \quad \delta_{T2} = a^2, \quad \delta_{T3} = a^3 .$$

The corresponding eigenvectors are

$$h_{T1}(x) = 1 - 0.1592x^4 - 0.5405x^8 + ... ,$$

$$h_{T2}(x) = 1 + 5.7050x^2 - 1.6908x^4 - 0.0806x^6 - 0.5774x^8 + ... ,$$

$$h_{T3}(x) = 1 + 9.2816x + 0.7026x^4 - 0.1312x^5 + ... \qquad (4)$$

The first and the second vectors are related to the even subspace while the third one has an odd part

$$h_{T3}(x) - h_{T3}(-x) \propto g'_T(x)/x^2 .$$

In accordance with the general RG method idea, the number of essential eigenvalues gives us the codimension of the critical situation. In other words, this is the number of parameters which the family of systems must have to demonstrate typically this situation. However, we explained at the beginning of the Letter that the tricriticality may have a codimension of either three (a quartic extremum in the initial map) or two (the situation of mapping one extremum to another). It can be found in the last case that some sort of *hidden symmetry* exists. So, perturbations of the RG fixed point, which appear when we change the map parameters, do not contain the third eigenvector.

The simplest model map demonstrating the codimension-three tricriticality is constructed by adding an odd term to the map (1):

$$x_{n+1} = 1 - Ax_n^2 - Bx_n^4 - Cx_n . \qquad (5)$$

The tricritical point is obtained as the limit of a period doubling cascade at a line $A=0$, $C=0$, where the map has a quartic extremum:

$$A=0, \quad B=1.5949013562288, \quad C=0 . \qquad (6)$$

Also the codimension-two tricriticality may be found in this map, but the simplest example of it is found in the cubic map

$$x_{n+1} = A - Bx_n + x_n^3 . \qquad (7)$$

The tricritical point is just the limit point of the period doubling cascade at the line $3A^2 = B(1-2B/3)^2$ where the condition of extremum-to-extremum mapping is valid. It is

$$A = -0.242698757265,$$

$$B = 1.951385777782 . \qquad (8)$$

The next example is the circle map, describing quasiperiodicity, phase locking and chaotization. It has the general form

$$x_{n+1} = x_n + kf(x_n) + r , \qquad (9)$$

where f is a function obeying $f(x+2\pi) = f(x)$. The most popular particular case is the sine-map

$$x_{n+1} = x_n + r + k \sin x_n . \qquad (10)$$

Inside the Arnold tongues in the k, r plane, where the phase locking takes place, the transition to chaos via period doubling bifurcations is observed typically when k is increased. Also the codimension-two tricritical points exist being the ends of the Feigenbaum critical lines [6]. One of them is

$$r=0.428467608369, \quad k=3.074701618846 . \quad (11)$$

In more general circumstances, the nonlinear function $f(x)$ may be controlled itself by some additional parameters. Taking it as a sum of two harmonics

$$f(x)=\sin x_n+b\sin(2x_n+\varphi) , \quad (12)$$

we obtain the map

$$x_{n+1}=x_n+r+k\,[\sin x_n+b\sin(2x_n+\varphi)] , \quad (13)$$

in which the codimension-three tricriticality may be found. We select a special parameter relation for the function $f(x)$ to have quartic extrema and calculate the limit of the period doubling cascade inside the Arnold tongue. For a particular r, $r=-0.576$, we have

$$k=3.792812600, \quad b=0.238132526,$$
$$\varphi=1.037954989 . \quad (14)$$

We emphasize one important universal property of the tricriticality. In the tricritical point the map has all period-2^n cycles, which are unstable and are characterized (in the large n limit) by the same universal multiplier $\mu_c=-2.05094049$. In fact, the map giving the evolution of the state over 2^n temporal steps looks like the universal function $g_T(x)$ for large n – it may differ only in scale. So the multiplier of the period-2^n cycle is a universal constant obtained as the derivative of $g_T(x)$ at the fixed point:

$$\mu_c=g'_T(x_*), \quad x_*=g_T(x_*) .$$

Let us turn now to two-dimensional maps and select the examples reducible to the above one-dimensional maps in the strong dissipation limit. The parameter responsible for including the second dimension will be denoted by D always. The strong dissipation limit corresponds to $D=0$.

The first two examples are the modified Hénon maps

$$x_{n+1}=1-Ax_n^2-Bx_n^4-Cx_n-Dy_n, \quad y_{n+1}=x_n , \quad (15)$$

and

$$x_{n+1}=A-Bx_n+x_n^3-Dy_n, \quad y_{n+1}=x_n . \quad (16)$$

The third example is the dissipative standard Zaslavsky map [7]

$$x_{n+1}=x_n+\nu+\mu\nu[Y_n+\varepsilon f(x_n)] ,$$
$$Y_{n+1}=e^{-\Gamma}[Y_n+\varepsilon f(x_n)] , \quad (17)$$

where ν, μ, ϵ, and Γ are parameters. We take the nonlinear function (12) and change the variables and parameters as follows,

$$y=e^{-\Gamma}x-\mu\nu Y, \quad k=\epsilon\mu\nu,$$
$$r=\nu(1-e^{-\Gamma}), \quad D=e^{-\Gamma} , \quad (18)$$

to obtain the map

$$x_{n+1}=x_n+r+k[\sin x_n+b\sin(2x_n+\varphi)]$$
$$+D(x_n-y_n), \quad y_{n+1}=x_n . \quad (19)$$

We know that the maps (15), (16), and (19) have tricritical points for $D=0$ when they are reduced to (5), (7), and (13), respectively. If we take such a tricritical point and make $D\neq 0$, some perturbation of the RG equation fixed point will appear. One can analyze this perturbation and reveal the contribution of all essential eigenvectors (4). With this aim, let us calculate the derivatives of the period-2^n cycle multipliers with respect to D at the tricritical point for different n. If the ith eigenvector contributes to the perturbation, the component, increasing as δ_{Ti}^n, will be present in the sequence $\partial\mu_n/\partial D$.

At first, let us consider the case when the codimension-two tricriticality is realized in the strong dissipation limit. In the first column of table 1 we give the values of $\partial\mu_n/\partial D$ calculated numerically for the map (16). We expect that this sequence is described by a relation

$$\partial\mu_n/\partial D=\sum_{i=1}^{3} C_i\delta_i^n . \quad (20)$$

Using the least-squares method, we have found

$$C_1=0.296624 , \quad C_2=0.546874 ,$$
$$C_3=-0.270258 . \quad (21)$$

Comparing both columns of table 1 one can see that the coincidence is excellent.

So we find that including the second dimension gives rise to the RG fixed point perturbation containing all three eigenvectors (4). This means that an attempt to find tricriticality in the two-dimen-

Table 1
Derivatives of the period-2^n cycle multipliers for the map (16) at the tricritical point $A=-0.242698757265$, $B=1.951385777782$, $D=0$.

n	$\partial\mu_n/\partial D$	
	direct calculation	by eqs. (20) and (21)
2	13.822108	13.901792
3	158.08801	157.86285
4	724.64118	724.74351
5	6899.6053	6899.0951
6	41195.446	41195.982
7	340319.34	340319.26
8	2274758.9	2274758.8
9	17528916.	17528916.

sional map near the considered point is condemned to failure: it is impossible to compensate the third eigenvector contribution by variation of other parameters. This is because the parameters present in the one-dimensional map influence only coefficients of the first and second eigenvectors due to the above-mentioned hidden symmetry.

We conclude that the tricritical point existing for $D=0$ has codimension three in the parameter space of the two-dimensional map. The third eigendirection associated with the scaling constant δ_3 is transversal to the plane $D=0$. So *the appearance of the second dimension destroys the hidden symmetry.* The same situation is realized in any other Hénon-like map if the extremum-to-extremum mapping takes place in the strong dissipation limit. One more example is given by the Zaslavsky map (17) with $f(x)=\sin x$, which is reduced to the one-dimensional sine-map for $D=e^{-\Gamma}\to 0$.

Let us turn now to the situation when the one-dimensional map obtained in the strong dissipation limit exhibits the codimension-three tricriticality. In this case any perturbation of the RG fixed point may be compensated by changing three parameters of the one-dimensional map, because it allows one to control all three essential eigenvectors. So, if the parameter D is increased, the tricritical point will simply move in the space of three other parameters.

We note, however, that the method of searching for tricriticality at the lines where the quartic extremum exists, cannot be generalized simply for the two-dimensional case. In fact, now it is difficult (if possible) to define a location and type of the point, which could play the same role as the one-dimensional map extremum. Hence, we use an alternative approach. We search for a point in the three-dimensional space, where the multipliers of three cycles with long periods 2^n, 2^{n+1}, and 2^{n+2} are equal to μ_c. It gives a good approximation for the tricritical point with precision increasing rapidly with n.

For the modified Hénon map (15) we start from the tricritical point (6). Increasing D we trace the codimension-three tricritical line in the four-dimensional parameter space (A, B, C, D) by the above explained technique. Some particular points are presented in table 2.

One can verify that all universal scaling properties intrinsic to the tricriticality are valid for the two-dimensional map (15) at the points given in table 2.

Figure 1 shows views of attractor at the tricritical point for $D=0.3$. Each succeeding picture gives a magnified part of the previous one. From one picture to another the magnification is increased by factor a_T. It can be seen that the Cantor-like structure reproduces itself in smaller scales in accordance with the expected tricritical scaling.

Due to dissipative nature of the map (15) with $D<1$, an approximate one-dimensional map may be found to describe its dynamics. If we consider the tricritical point, this map must have a quartic extremum. Figure 2 shows the numerically found maps for the model (15) with $D=0.3$. Random initial conditions were taken, then a number of preliminary iterations was done to fit to a central manifold, and, at last, the points x_{n+N} versus x_n were plotted for $N=1, 2, 4$. The shapes looking like quartic extrema in fact can be seen in the pictures.

In fig. 3 the power spectra are compared for time series generated by two- and one-dimensional maps at their tricritical points. A remarkable coincidence of the spectra supports the tricritical universality. Note that the tricritical spectrum differs from the ordinary spectrum at the period doubling accumulation point: the rate of the subharmonic amplitude decreasing from one level to another is about 10 dB rather than the convenient value of 13.4 dB.

Finally, one can verify the scaling properties of the parameter space near the tricritical point. With this aim let us calculate the derivatives of multipliers with

Table 2
The codimension-three tricritical points for the two-dimensional maps.

Modified Hénon map (16)				Modified Zaslavsky map (19). $R=-0.576$			
A	B	C	D	k	b	φ	D
0.00765746	2.10701337	0.03496330	0.1	4.04724523	0.23578705	0.99083357	0.1
0.01447916	2.76632514	0.06465294	0.2	4.31254195	0.23390781	0.95416022	0.2
0.02092622	3.60380862	0.08986689	0.3	4.58698778	0.23241654	0.92590326	0.3

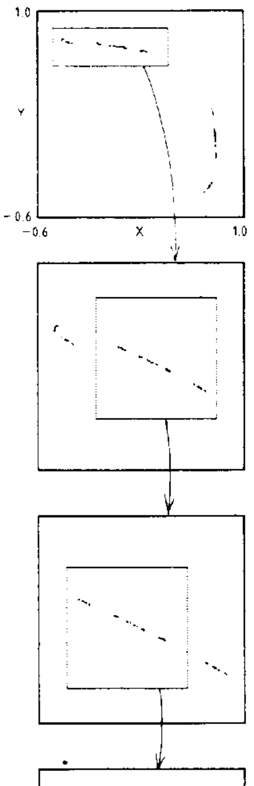

Fig. 1. Tricritical attractor for the modified Hénon map (16), $D=0.3$, other parameters are given in table 2. The property of selfsimilarity is demonstrated: each succeeding picture shows a magnified part of the previous one.

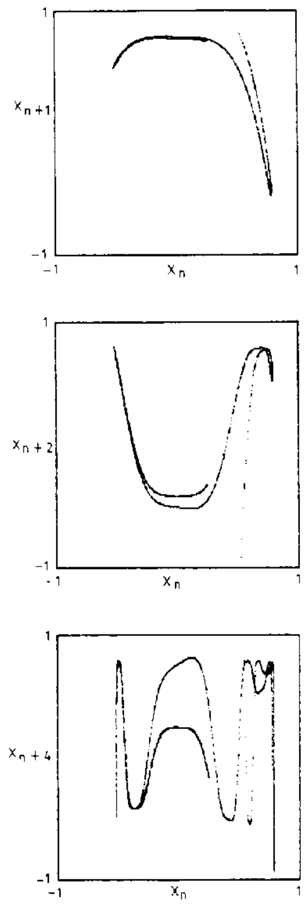

Fig. 2. Plots of one-dimensional maps over one, two and four iterations obtained numerically for the system (16) at the tricritical point for $D=0.3$.

respect to the parameters of the tricritical point for the period 2^n, 2^{n+1}, 2^{n+2}, 2^{n+3} and find the matrices

$$\hat{\mathbf{M}}_n = \begin{pmatrix} \partial\mu_n/\partial A & \partial\mu_n/\partial B & \partial\mu_n/\partial D \\ \partial\mu_{n+1}/\partial A & \partial\mu_{n+1}/\partial B & \partial\mu_{n+1}/\partial D \\ \partial\mu_{n+2}/\partial A & \partial\mu_{n+2}/\partial B & \partial\mu_{n+2}/\partial D \end{pmatrix} \qquad (1)$$

and $\hat{\mathbf{M}}_{n+1}$. If the perturbation vector $\boldsymbol{r}=(\Delta A, \Delta B, \Delta C)$ corresponds to an eigendirection in the parameter space with a scaling constant δ_{Ti}, we have, evidently,

$$\hat{\mathbf{M}}_{n+1}\boldsymbol{r}=\delta_{Ti}\hat{\mathbf{M}}_n\boldsymbol{r}\,.$$

Thus, the constants δ_{Ti} will be obtained as the eigenvalues of the matrices $\hat{\mathbf{M}}_n^{-1}\hat{\mathbf{M}}_{n+1}$ for sufficiently

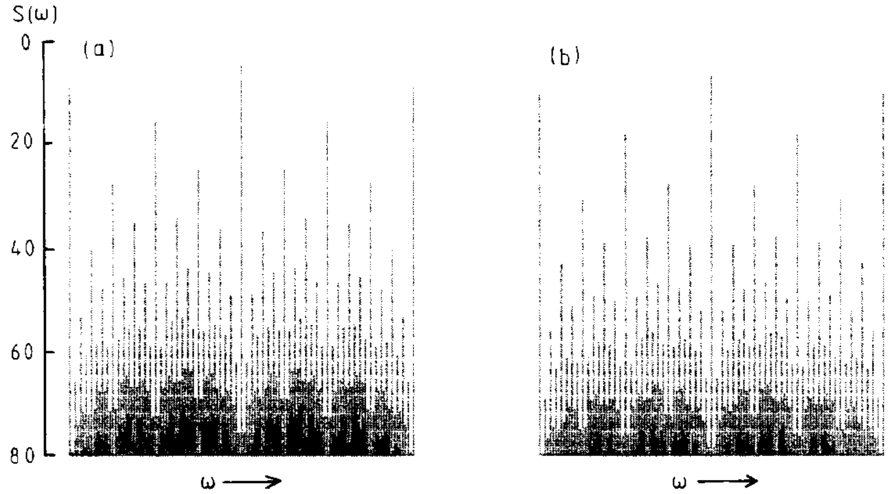

Fig. 3. Power spectra of time series generated by two- and one-dimensional maps at their tricritical points: (a) map (1), $A=0$, $B=1.594901356$; (b) map (16), $D=0.3$, other parameters are given in table 2.

large n. The calculated eigenvalues for the map (15) are presented in table 3 for different n. One can see that their values agree with δ_{Ti} found from the RG analysis.

Let us turn now to Zaslavsky map (19) and start from the tricritical point (14), $D=0$. We fix one of the parameters, $r=-0.576$, and increase D changing the remaining parameters k, b, φ to keep the tricritical situation. As a result, we obtain a tricritical line in (k, b, φ, D)-space and the points presented in table 2 for particular D's. A verification of different scaling properties supports the tricritical universality in this case too. For example, the eigenvalues of matrices $\hat{\mathbf{M}}_n^{-1}\hat{\mathbf{M}}_{n+1}$, where

$$\hat{\mathbf{M}}_n=\begin{pmatrix} \partial\mu_n/\partial k & \partial\mu_n/\partial b & \partial\mu_n/\partial\varphi \\ \partial\mu_{n+1}/\partial k & \partial\mu_{n+1}/\partial b & \partial\mu_{n+1}/\partial\varphi \\ \partial\mu_{n+2}/\partial k & \partial\mu_{n+2}/\partial b & \partial\mu_{n+2}/\partial\varphi \end{pmatrix}$$

for the map (19), also converge to the universal numbers δ_{Ti}.

We conclude that in contrast to Feigenbaum universality which takes place in one-parameter families of nonlinear dissipative systems of arbitrary dimension, the tricriticality appears in a different manner in one- and multi-dimensional cases.

In one-dimensional maps there are two types of tricriticality: (i) the codimension-three tricriticality arising in unimodal maps, when an extremum becomes quartic, (ii) the codimension-two tricriticality, when one quadratic extremum is mapped to another. As we find, in two-dimensional Hénon-like maps and, evidently, in other complicated systems – maps or differential equations – three parameters are needed to realise the tricriticality.

Table 3
Eigenvalues of matrices $\hat{\mathbf{M}}_n^{-1}\hat{\mathbf{M}}_{n+1}$. Modified Hénon map (16), $A=0.02092622$, $B=3.60380862$, $C=0.08986689$, $D=0.3$. Modified Zaslavsky map (19), $R=-0.576$, $k=4.58698778$, $b=0.23241654$, $\varphi=0.92590326$, $D=0.3$.

n	Modified Hénon map (16)			Modified Zaslavsky map (19)		
	δ_1	δ_2	δ_3	δ_1	δ_2	δ_3
1	7.0690	3.2314	−4.4507	6.9182	3.3795	−5.0268
2	7.2001	2.9509	−4.8495	7.2863	2.8468	−4.8429
3	7.2841	2.8254	−4.8251	7.2709	2.8712	−4.8298
4	7.2586	2.7992	−4.7938	7.2843	2.8092	−4.8260
5	7.2989	2.6028	−4.9069	7.2650	2.1720	−4.8177

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