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New types of critical dynamics for two-dimensional maps

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New universality classes are discovered for critical phenomena which may be observed in nonlinear systems during a multiparameter study of roads to chaos. A sketch of their classification is discussed.

1. Up to now several scenarios of transition to chaos in dynamical systems observed under the change of some control parameters have been revealed and studied in detail [1]. In dissipative systems these are the Feigenbaum period doubling sequence [2], transitions via intermittence [3] and via quasiperiodicity [4]. The renormalization group (RG) analysis has been developed, showing that each of these scenarios is connected with their respective universality class [2,4,5]. All dynamical systems relating to the same class possess the same set of critical indices featuring the behaviour in the critical situation (exactly at the border of chaos) and in its vicinity. One-dimensional maps, the simplest systems demonstrating the above scenarios, are considered now as the canonical representatives of their universality classes.

While studying the systems with a larger number of dimensions and control parameters one may expect to meet some other types of critical behaviour. Suppose, for example, that in the parameter space of some multi-dimensional system there are period doubling bifurcation surfaces accumulating to the critical surface in accordance with the Feigenbaum law. Moving along this surface (with codimension one) one may come to the boundary – the critical surface with codimension two (its appearance may be caused by a new mode stability loss). In turn, it may have, as a boundary, the critical surface with codimension three and so on. Each type of critical behaviour in this hierarchy must relate to its own universality class. The problem arises, to develop RG analysis covering these hypothetical types of critical behaviour, to find their universal critical indices and to classify the types of critical dynamics in increasing codimension order in rough analogy to bifurcation and catastrophe theory [6].

In this Letter we consider, from this point of view, the critical phenomena occurring in two-dimensional maps and take into consideration several new universality classes which may be observed in nonlinear systems of diverse nature under the multi-parametric study of transition to chaos.

2. We begin with a two-dimensional generalization of the Feigenbaum RG procedure. Having the two-dimensional map

$$G_0:\quad X_{n+1}=g_0(X_n, Y_n)\,,\qquad Y_{n+1}=f_0(X_n, Y_n)\,, \tag{1}$$

we repeat it twice and then use the linear transformation of the dynamical variables S to make the new map $G_1=S^{-1}G_0G_0S$ presenting the two-time-step evolution as similar to the initial one as possible. It is convenient to choose the coordinate system in the phase plane X, Y in such a way that the above linear transformation would have the diagonal form,

$$S:\quad X\to X/a\,,\quad Y\to Y/b\,.$$

The multiple repetition of this procedure leads to the

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recurrent RG equations: $G_{k+1}=S^{-1}G_kG_kS$ or in explicit form:

$$g_{k+1}=ag_k(g_k(X/a,\ Y/b),f_k(X/a,\ Y/b))\ ,$$
$$f_{k+1}=bf_k(g_k(X/a,\ Y/b),f_k(X/a,\ Y/b))\ . \qquad (2)$$

In accordance with the general RG analysis idea, each saddle fixed point (or saddle cycle) of this functional transformation is responsible for a definite type of critical behaviour and universality class. To find such a solution means at the same time to find the scale factors a and b.

The eigenvalue spectrum of the RG transformation (2) linearized near the fixed point (or cycle) is of particular importance. The number of eigenvalues which are greater than unity in modulus and are not connected with the infinitesimal dynamical variable changes defines the codimension – the number of parameters needed to meet this critical behaviour typically. The eigenvalues themselves are the factors of scaling along the appropriate directions in the parameter space.

To find the fixed points and cycles of the RG transformation numerically one may approximate the functions g and f through finite orthogonal polynomial expansions. This allows one to reduce the functional equations (2) to the set of nonlinear algebraic equations. The latter may be solved then by the Newton technique.

Eq. (2) covers a lot of well-known types of critical dynamics. For instance, taking $f\equiv 0$, we obtain the equation [2]

$$g_{k+1}(X)=ag_k(g_k(X/a))$$

for the function $g(X)\equiv g(X,\ 0)$, covering, depending on some extra conditions, the Feigenbaum (codimension one) [2], intermittency (codimension one) [5], tricritical (codimension two) [7] fixed points. We shall denote them as F, I and T, respectively.

Eq. (2) has also the fixed point H describing the critical dynamics of Hamiltonian systems at the borderline of chaos via the period doublings studied with the help of the RG technique by several authors [8]. We have reproduced the calculations of this fixed point by numerically solving eq. (2) and have found that besides earlier known values $\delta_{H,1}=8.72109720$ and $\delta_{H,2}=2$ [8,9], there are no other essential eigenvalues in the linearized RG operator spectrum. The perturbations of the fixed point related to $\delta_{H,1}$ leave the map in the area preserving class while the perturbations related to $\delta_{H,2}$ are responsible for the dissipation. So, in the space of common two-dimensional maps the critical behaviour H has codimension two and may occur during the two-parameter analysis.

3. The main problem arising while solving eq. (2) and, therefore, searching for new types of critical behaviour consists really in choosing the initial approximation for the Newton method. We use here a "going down the codimension" technique, taking the case of a function g independent of the second argument for a starting point because of its relative simplicity for studying. The corresponding type of critical dynamics arises in the system of two unidirectionally coupled period doubling subsystems when they are in strict succession brought just onto the border of chaos by choosing their own control parameters. In ref. [10] this situation was called *bicritical* and studied empirically. It may be described by the model map

$$F:\quad x_{n+1}=1-\lambda x_n^2\ ,\quad y_{n+1}=1-Ay_n^2-Bx_n^2\ , \qquad (3)$$

where λ, A, B are parameters. The first equation is independent of the second one and the x component undergoes period doublings at the known parameter values of $\lambda_k=0.75,\ 1.25,\ 1.3681,\ ...$. Suppose we take some $B>0$. If we start from small enough A for $\lambda=\lambda_k$ in the parameter plane $(\lambda,\ A)$ one of the period-2^k cycle multipliers of the map (3) is -1 and the second one is near zero. Increasing A along the line $\lambda=\lambda_k$ we may come to the terminal point $(\lambda_k,\ A_k)$ where the second multiplier is -1 too. It means a new mode becomes unstable. Estimation of the limit for the terminal point sequence $(\lambda_k,\ A_k)$ gives the bicritical point $(\lambda_c,\ A_c)$. For the particular $B=0.375$ we have $\lambda_c=1.401155189$, $A_c=1.124981403$.

Further, we may use the map (3) to get an approximation for the fixed point functions $(g_B,\ f_B)$. For the first, we make $N=2^k$ iterations of the point (0, 0) and find the scale factors $\tilde{X}=x_N$, $\tilde{Y}=y_N$. Suppose then

Table 1
Fixed point B: polynomial approximation for the universal function $f_B(x, y)$.

	1	x^2	x^4	x^6	x^8
1	1.000000	−0.596905	−0.032157	0.018457	−0.000201
y^2	−0.855639	−0.302943	0.054630	0.021499	−0.004860
y^4	−0.431738	0.087452	0.091136	−0.011023	−0.003242
y^6	0.087486	0.180356	0.009298	−0.031914	0.005042
y^8	0.152662	0.060337	−0.096310	0.017439	0.000000
y^{10}	0.060864	−0.153737	0.037690	0.000000	0.000000
y^{12}	−0.101867	0.047570	0.000000	0.000000	0.000000
y^{14}	0.026310	0.000000	0.000000	0.000000	0.000000

$$\begin{pmatrix} g_B^{[N]}(X,Y) \\ f_B^{[N]}(X,Y) \end{pmatrix} = \begin{pmatrix} 1/\tilde{X} & 0 \\ 0 & 1/\tilde{Y} \end{pmatrix} (F)^N \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \qquad (4)$$

Note that the calculation results for $N=32, ..., 256$ are in good agreement.

With the above procedure we may obtain the values of $g_B^{[N]}$ and $f_B^{[N]}$ at the points of some net, approximate them by the orthogonal polynomial expansions and then take them as an initial point for the Newton method, which converges strongly. As a result, we have two functions: g_B and f_B. The first one is, obviously, the well known Feigenbaum function [1,2]: $g_B(X, Y) \equiv g_F(X)$. The polynomial approximation for the function $f_B(X, Y)$ is presented in table 1. The scaling factors a and b, together with the relevant eigenvalues of the linearized RG operator (2), are listed in table 2. The eigenvalues $\delta_{B,1}$ and $\delta_{B,4}$ are related to the fixed point perturbations preserving unidirectionality of coupling for the map (3) [11], and the others, $\delta_{B,2}$ and $\delta_{B,3}$, are responsible for the appearance of the contradictional coupling. So, the complete codimension for bicriticality is four.

4. We introduce now an additional term in the map (3), including the perturbation related to the $\delta_{B,2}$ eigenvalue, and consider the map

$$F': \quad x_{n+1} = 1 - \lambda x_n^2 - C y_n^2 , \qquad y_{n+1} = 1 - A y_n^2 - B x_n^2 , \qquad (5)$$

Taking some B, C and small A, one may observe the Feigenbaum period doublings with λ increasing. We can find the bifurcation point numerically and then move along the bifurcation curve towards increasing A until the modulus of the second multiplier becomes unity too. We expect that with increasing cycle period the found terminal points would accumulate to some limit (λ_c, A_c) and it would be the critical point of a new type. The calculations show that for

Table 2
Universal scaling factors and eigenvalues of the linearized RG operator for considering types of critical behaviour.

	a, b	δ	Codimension
fixed point B	$a=-2.502907876$	4.66920161	4
	$b=-1.505318159$	4.2968970	
		−4.16161049	
		2.39272443	
fixed point FQ	$a=-4.008157849$	6.32631925	3
	$b=-1.900071670$	3.44470967	
		−1.90007167	
RG period-2 cycle C	$a^2=6.565349940$	92.43126348	2
	$b^2=22.120227422$	4.19244418	
fixed point F	$a=-2.502907876$	4.66920161	1

Table 3
Fixed point FQ: polynomial approximations for the universal functions $g_{FQ}(x, y)$ and $f_{FQ}(x, y)$.

	1	x	x^2	x^3	x^4	x^5
1	1.000000	0	−0.000003	0	0.000003	0
	1.000000	0	−0.000003	0	0.000002	0
y	0	−2.796050	0	0.000233	0	−0.000085
	0	−0.711423	0	0.000005	0	0.000000
y^2	0.066947	0	0.210253	0	−0.000092	0
	−1.097926	0	0.001860	0	−0.000025	0
y^3	0	1.362135	0	−0.005578	0	0.000688
	0	0.086519	0	0.007119	0	0.000000
y^4	1.541702	0	−0.158674	0	−0.003899	0
	0.157331	0	0.037350	0	−0.000294	0
y^5	0	−0.816413	0	−0.016455	0	−0.000362
	0	0.043877	0	−0.003629	0	0.000000
y^6	−1.061435	0	−0.007651	0	0.009521	0
	−0.018284	0	−0.013299	0	0.000000	0
y^7	0	0.179277	0	0.033055	0	0.000000
	0	−0.017820	0	0.000519	0	0.000000
y^8	0.197242	0	0.065557	0	−0.004133	0
	−0.005775	0	0.002289	0	0.000000	0
y^9	0	−0.003142	0	−0.010749	0	0.000000
	0	0.003437	0	0.000000	0	0.000000
y^{10}	−0.012557	0	−0.018792	0	0.00000	0
	0.001913	0	0.000000	0	0.00000	0

small negative C they do converge. The best estimates for parameter values $B=0.375$, $C=-0.25$ are $\lambda_c=1.654524590$, $A_c=1.030837593$. It is the end point of the Feigenbaum critical line F. Our computations show also that one may observe quasiperiodicity arbitrarily close to this point. So, we denote it as FQ.

We may use now the map (5) at the found point to get approximations for the functions g_{FQ} and f_{FQ} corresponding to the new fixed point of the RG equation (2). A difficulty in comparison to the previous case consists in the lack of coincidence between the scaling directions in the phase space and the coordinate axis x, y. Therefore we must turn to new variables (X, Y), appropriate for the RG equations. Note that the eigenvectors of a derivative matrix calculated for the cycle element nearest to the origin of the period-2^k cycles tend asymptotically to the vector (1, 1.1443) with increasing k. This direction we choose as a new Y axis. The X axis direction is not essential [#1], so we keep the old one. As a result of the phase space transformation $x=X+Y/1.1443$, $y=Y$, the map (5) becomes

$$\tilde{F}':\quad X_{n+1}=0.126103-1.326813X_n^2-2.318996X_nY_n+0.137564Y_n^2\,,$$

$$Y_{n+1}=1-0.375X_n^2-0.655422X_nY_n-1.317223Y_n^2\,. \qquad (6)$$

From this point we can act just like in the previous case using relations (4) to obtain the functions $g_{FQ}^{[N]}(X, Y)$ and $f_{FQ}^{[N]}(X, Y)$. The calculation results for $N=32$, 64, 128, are again in good agreement and substituting them into the Newton procedure we obtain the polynomial approximations for g_{FQ} and f_{FQ} presented in table 3. Note that both functions are in-

[#1] The reason for choosing the second coordinate axis to be irrelevant consists in the fact that the change of variables corresponding to a small turning of this axis is related to the linearized RG equation eigenvalue $b/a\approx 0.47<1$. Therefore, the perturbation associated with small axis turning decreases strongly with increasing the number of iterations used to get estimates of the universal functions.

variant under inversion ($X\to -X$, $Y\to -Y$). The best estimates of the scaling factors a, b and relevant eigenvalues of the linearized RG operator δ_{FQ} are listed in table 2.

5. The complete codimension of the FQ point is three. The perturbations related to the first two eigenvalues $\delta_{FQ,1}$ and $\delta_{FQ,2}$ preserve the inversion invariance, and the third one $\delta_{FQ,3}$ leads out of this class. We expect that the addition of the latter type term to the map (5) would cause the appearance of another type of critical behaviour with codimension two and, furthermore, it would be a period-2 cycle of the RG transformation rather than a fixed point because of the negative $\delta_{FQ,3}$ value. We consider next the map

$$F'':\quad x_{n+1}=1-\lambda x_n^2-Cy_n^2+\epsilon x_n\,,$$
$$y_{n+1}=1-Ay_n^2-Bx_n^2\,. \tag{7}$$

One can reproduce for parameter values $B=0.375$, $C=-0.25$, $\epsilon=-0.12$ the procedure developed in the previous sections and move along the bifurcation lines in the parameter plane (λ, A) until a new mode becomes unstable [#2]. Defining the critical point (λ_c, A_c) as a terminal points sequence limit we obtained $\lambda_c=1.5814935557 45$, $A_c=1.016156060448$. Further, one can try to get the initial approximations for the universal functions g and f relating to the period-2 solution of the RG equations with the use of map (7) at the critical parameter point but new problems arise in this way.

In the first place, the similarity center (point $X=0$, $Y=0$ in the RG equation scaling coordinates) does not coincide with the origin in the natural phase variables plane and the procedure must involve the technique for its estimation. In the second place, approximations obtained for a reasonable (of the order of 4096) number of iterations do not hit the attraction basin of the Newton method. A special procedure was elaborated to make the solution more precise, based on the construction of consequent polynomial approximations. At each step of the procedure the expansion factors related to X^2 for the function g and to Y for the function f are taken as parameters, and the critical point is defined as a bifurcation lines terminal point sequence limit in this parameter plane. The scaling center position is defined by the location of period-1, -4, -16 cycle elements calculated at the critical point. Then the fourfold iteration and rescaling (just like in (4)) is done to find the next polynomial approximation and so on. It is clear that this procedure is suitable not for the RG cycle case only but for the fixed point too. By multiple repetition of these calculations the attraction basin of the Newton method was reached. The resultant polynomial approximations for the functions g and f corresponding to one of the RG period-2 cycle steps are presented in the table 4. The procedure to make the solution more precise was carried out until we were sure that its limit functions coincided with the results of the Newton method. Note that we use the normalizing conditions in the form $g(0, 0)=1$, $f(0, 0)=0.1$ for computational reasons.

The found type of critical behaviour is the period-2 cycle of the RG equation and it differs from the other known universality classes strongly. We shall denote it as C (cycle). In this case the reproduction of the structure of the parameter and phase space in smaller and smaller scales (scaling) takes place not under doubling but under quadrupling of the period. The scale factors and relevant eigenvalues, listed in table 2, are comparatively large since they are defined over the RG cycle period. The codimension of the universality class is two, so, the further way "down the codimension" leads to the known types of critical behaviour F and I. Our computations confirm that the Feigenbaum and intermittency critical lines intersect at the point C.

6. The relationship between different types of critical behaviour (universality classes) may be illustrated by the following diagram:

```
H(2)
    ↘
     F(1)←C(2)←FQ(3)←B(4) ,
    ↗         ↓
T(2)         I(1)
```

where the designations introduced above are used and the figures in brackets indicate the codimension.

[#2] As we found, for large enough ϵ the described procedure of searching the Feigenbaum surface border for the system (7) may also lead to critical behaviour of type H (see section 2). For instance, for $B=0.375$, $C=-0.25$, $\epsilon=0.2$ the critical point H is ($\lambda_c=1.759302100$, $A_c=1.051352159$).

Table 4
Period-2 cycle of RG equations C: polynomial approximations for the universal functions $g_C(x, y)$ and $f_C(x, y)$.

	1	x^2	x^4	x^6	x^8
1	1.000000	−1.529231	0.019299	0.044942	−0.004036
	0.100000	−0.165914	−0.144840	0.041494	−0.000822
y	2.314272	−0.592293	−0.245794	0.046360	−0.001546
	1.349157	0.221319	−0.278699	0.014219	0.005468
y^2	0.999075	0.380929	−0.178697	0.010370	0.002284
	0.014409	0.555718	−0.102917	−0.027465	0.006626
y^3	−0.131319	0.289243	−0.045086	−0.006058	0.001836
	−0.324387	0.228667	0.037282	−0.020910	0.000000
y^4	−0.162450	0.074830	0.004415	−0.004505	0.000000
	−0.156972	−0.009586	0.040148	−0.002696	−0.002532
y^5	−0.044384	0.002051	0.008018	−0.001478	0.000000
	−0.012139	−0.035864	0.009577	0.003112	0.000000
y^6	−0.004093	−0.005277	0.002278	0.000000	0.000000
	0.012352	−0.011751	−0.003514	0.002904	0.000000
y^7	0.001581	−0.001742	0.000000	0.000000	0.000000
	0.004374	0.000666	−0.002623	0.000000	0.000000
y^8	0.000657	−0.000276	0.000000	0.000000	0.000000
	−0.000248	0.002362	−0.001140	0.000000	0.000000
y^9	0.000000	0.000000	0.000000	0.000000	0.000000
	−0.000321	0.000641	0.000000	0.000000	0.000000

The arrow joining two symbols means that the critical surface of the first type may be the border of the critical surface of the second type with the codimension being one less. Obviously, in an arbitrarily small neighbourhood of a definite critical point, all types of critical behaviour are realized, which can be reached from the corresponding diagram point by moving along the arrows. We believe that the supposed diagram may serve as a rough sketch of the critical dynamics types classification scheme.

We discuss now the possibility of experimental realization of new types of critical behaviour. In spite of high codimension, the bicriticality B has been recently observed in electronic experiments [10] in the system of two periodically driven nonlinear oscillators, since the unidirectional coupling was easily carried out with the use of a special amplifier. It is likely that C and FQ types of critical behaviour may be realized in similar systems with mutual coupling. Formally, the codimensions of C and FQ are two and three, respectively, and they should be searched for by the two- and three-parameter analysis of the transition to chaos while moving along the Feigenbaum critical surface. It should be noted, however, that the scaling featuring the C point is hardly to be observed because of the presence of the linearized RG operator eigenvector with eigenvalue only slightly less than unity ($\delta_{C,3}=0.93$). In practice, it means that in spite of codimension two, three control parameters are desirable, the additional one to be used for removing the above slowly decreasing component.

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