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VARIETY OF TYPES OF CRITICAL BEHAVIOR AND MULTISTABILITY IN PERIOD-DOUBLING SYSTEMS WITH UNIDIRECTIONAL COUPLING NEAR THE ONSET OF CHAOS

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The dynamics of two unidirectionally coupled period-doubling systems is investigated depending on three relevant parameters (control parameters of subsystems and coupling). There is a hierarchy of critical behavior types. Feigenbaum's critical surfaces existing in the parameter space are bounded by tricritical lines and intersect along the bicritical line. These lines, in turn, intersect at a new multicritical point *BT*. Universality and scaling properties for all the critical situations are discussed, and the table of critical indices is given.

1. Introduction

It is commonly recognized that the question about scenarios of transition to chaos plays a fundamental role in nonlinear science. The question is: What are typical bifurcation sequences observed under slow varying control parameters, making the nonlinear system go from regular to chaotic behavior? As is generally known, a few scenarios have been revealed by the works of many investigators: transition to chaos via period-doubling cascade [Myrberg, 1963; Sharkovsky, 1964; Metropolis *et al.*, 1973; May, 1976], via intermittence [Afraimovich & Shil'nikov, 1974; Pomeau & Manneville, 1980], and via quasi-periodicity [Arnold, 1978; Ruelle & Takens, 1971; Shenker, 1982].

It is remarkable that the question about scenarios of transition to chaos involves not only qualitative answers. Indeed, it was discovered that quantitative universality and scaling properties of some sort are shown by nonlinear systems near the onset of chaos. The renormalization group (RG) approach was adopted for its theoretical foundation, similar to that developed earlier in the phase transition theory. This principal step was first made by Feigenbaum [1978, 1979], who considered the period-doubling systems. Later, analogous approaches were developed for cases of intermittence [Hirsh *et al.*, 1982; Hu & Rudnik, 1982] and quasiperiodicity [Feigenbaum *et al.*, 1982; Ostlund *et al.*, 1983].

In phase transition theory, the RG approach is applied to describe the critical behavior of a substance near the phase transition point (critical point), when the fluctuations with large spatial scales, essentially exceeding the interatomic distance, are presented. Similarly, we can speak about critical phenomena in nonlinear systems having in mind the dynamics near the onset of chaos when large temporal scales are presented, essentially exceeding all other characteristic times of the system. Using this terminology implies that the approach is well supported, which advances the concepts of universality, scaling and RG analysis in favour of qualitative descriptions of the bifurcation sequences.

In the families of nonlinear systems, depending on several parameters, new types of critical behavior and universality classes may appear, allowing the RG analysis (some examples are known [Chang *et al.*, 1981; Vul *et al.*, 1984; Zisook, 1984]). Such situations may be called *multiparameter criticality.* Each type of multiparameter criticality is characterized by the intrinsic unique structure of the parameter space near the critical point having a property of self-similarity. These structures may be considered as a generalization of the concept of scenarios for the multiparameter cases.

In this paper we consider a system of two unidirectionally coupled period-doubling maps in the context of multiparameter criticality. One can find that the codimension-1 critical surfaces exist in the three-dimensional parameter space where the Feigenbaum critical behavior takes place.[1] Moving transversely to these surfaces, one observes the standard period-doubling cascade. Moving along the critical surfaces, one can meet the critical lines of codimension 2. In turn, these lines have the codimension-3 critical points at their ends. RG analysis, universality, and scaling corresponding to these lines and points are discussed. Every critical situation is characterized by the definite, specific fractal attractor, power spectrum, and Lyapunov exponent dependence on the control parameters. Some of the considered critical behavior types are found to be connected with the appearance of multistability. It provides an opportunity to study and classify the multistable states in the context of critical phenomena, taking into account the universality and scaling properties.

2. Multistability. Double Feigenbaum's Point

A model system we are going to study is composed of two logistic maps:

$$x_{n+1} = 1 - \lambda x_n^2\,, \quad y_{n+1} = 1 - Ay_n^2 - Bx_n^2\,, \quad (1)$$

where x and y are dynamical variables characterizing the states of the first and the second subsystems, λ and A are control parameters, and B is a coupling parameter.

Coupling in Eqs. (1) is represented by the term Bx_n^2. The suggestion that the coupling is defined by a linear term gives the same results. Indeed, if the model map is chosen in the form

$$x_{n+1} = 1 - \lambda x_n^2\,, \quad y_{n+1} = 1 - ay_n^2 + \varepsilon x_n\,,$$

then, assuming $X_n = x_{n-1}$, $Y_n = y_n/(1+\varepsilon)$, $A = a(1+\varepsilon)$, $B = \varepsilon\lambda/(1+\varepsilon)$, we obtain exactly Eqs. (1) for the variables X_n, Y_n.

The system (1) is characterized by three parameters λ, A, B. So, the investigation of its dynamics consists in analyzing the three-dimensional parameter space topography.

If $B = 0$, the system (1) would break down into two uncoupled Feigenbaum's systems demonstrating period-doubling cascade depending on λ or A, respectively. If we choose suitable parameter values, each subsystem would have one stable period-2^k cycle. Thus, the composite system will have 2^k states differing one from another by a phase shift between oscillations in subsystems measured in the whole number of discrete time steps. These regimes are modified but remain stable when the coupling is introduced, at least while the B value is small enough. Hence, the regions exist in the parameter space (λ, A, B) where the system (1) has a number of attractors, or, in other words, the regions of multistability.

Let us fix the λ parameter and consider the second system behavior depending on A and B. As we have explained just now, the multistability is intrinsic to our system. It could be understood and recognized better, if we think about a foliated (A, B)-surface rather than about the (A, B)-plane. Each sheet of the surface would associate with one of the attractors. Overlapping sheets [in projection onto the (A, B)-plane] correspond just to the presence of the multistability.

We can get a preliminary perception of the foliated parameter surface by tracing its formation while increasing λ.

While λ is less than the first bifurcation value $\lambda_0 = 0.75$, the stable fixed point is realized in the first subsystem. The second subsystem is described simply by the logistic map which is reduced to the standard form $Y \to 1 - \lambda_{\text{eff}} Y^2$, $\lambda_{\text{eff}} = A(1 - B(2\lambda + 1 - \sqrt{1+4\lambda})/2\lambda^2)$ by the trivial variable change. It does not demonstrate multistability. Period-doubling bifurcations occur in the second subsystem at the lines which are defined by an equation $\lambda_{\text{eff}}(A, B) = \lambda_k$, $\lambda_k = 0.75, 1.25, 1.3681, \ldots$

When λ becomes greater than λ_0, the first system undergoes the first period-doubling bifurcation.

[1] In mathematics, the term "critical point" often designates a value of argument x at which the derivative of the function $f(x)$ vanishes. For functions of two or three arguments, they speak about critical lines or surfaces, having in mind the lines or surfaces in (x, y)- or (x, y, z)-space, where the Jacobian determinant is zero. Our terminology is related rather to the objects in the parameter space. Particularly, it was already introduced by physicists [Chang *et al.*, 1981; Geisel *et al.*, 1981; Shenker & Kadanoff, 1982; Fraser & Kapral, 1984; Bezruchko *et al.*, 1986] and seems suitable and useful because it underlines the methodologically important and deep analogy to phase transition theory. Unfortunately, some confusion of terminology is inevitable.

Starting with this moment, the cusp ${}^1C_{2^1}$ is located at the point $(\lambda_0, 0)$ in the (A, B) plane [see Fig. 1(a)]. Two lines of tangent bifurcations, fold lines ${}^1f_{2^1(0)}$ and ${}^1f_{2^1(1)}$, meet at this point. The partially overlapping sheets forming the shown surface correspond to the period-2 cycles; three cycles exist in the region of overlapping. Only two of the sheets are relevant, the facing one and the back one. They are associated with the period-2 cycles being stable in a region near the cusp point. If one goes along the (A, B)-plane from the right to the left above the cusp, then the jump from one sheet to another will take place when crossing the left fold line ${}^1f_{2^1(0)}$. The reverse jump will occur when crossing the right fold line ${}^1f_{2^1(1)}$ in the opposite direction.

Moving off the cusp point along the sheets shown in Fig. 1(a), one can observe the stability loss of the "parent" period-2 cycle and further bifurcations. They are accompanied by the appearance of more complicated attractors. However, these attractors may be associated with the sheet (at least, while this attractor does not merge with an attractor from another sheet).

A bifurcation diagram for the second subsystem is shown in Fig. 2 to illustrate the above statement. The values of y generated by the system are plotted versus the A parameter for $L = 0.85$ and $B = -0.1$. The diagram was obtained by iterating some ensemble of initial points distributed along y axis in order to visualize all existing attractors. It may be seen that the diagram is a composition of two overlapped "Feigenbaum's trees." Each of them corresponds to one of the sheets shown in Fig. 1(a). Note that the period-doubling cascade of the second subsystem begins from the period-2 cycle because

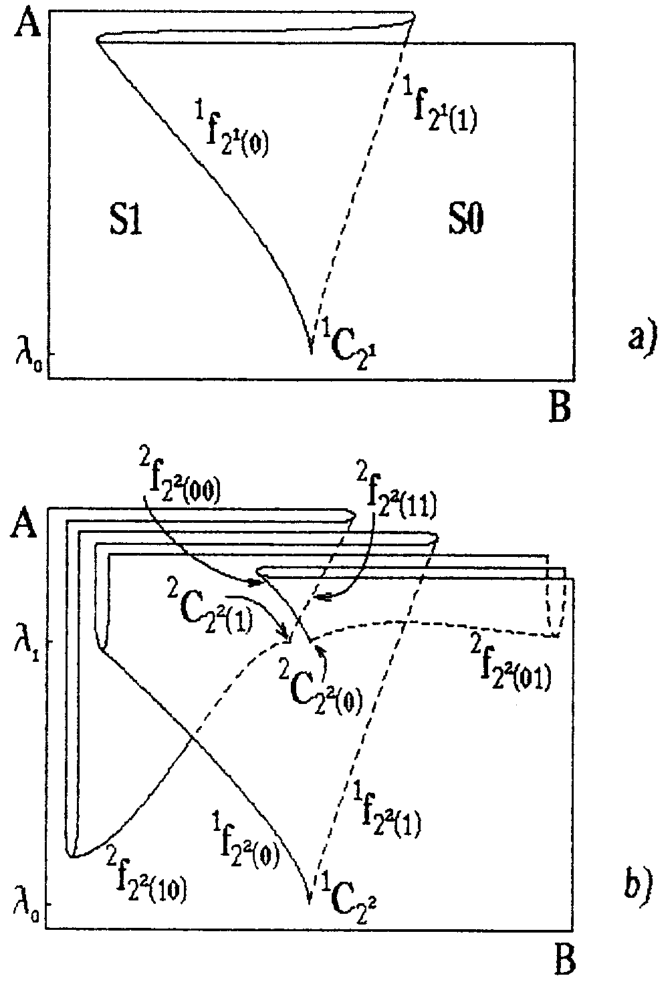

Fig. 1. Foliated structure of the (A, B) parameter plane for the second subsystem: (a) period-2 cycle in the first subsystem, (b) period-4 cycle in the first subsystem. The cusp points ${}^2C_{2^2(00)}$ and ${}^2C_{2^2(01)}$ have the same (A, B) coordinates but belong to different sheets. S0 is the in-phase sheet.

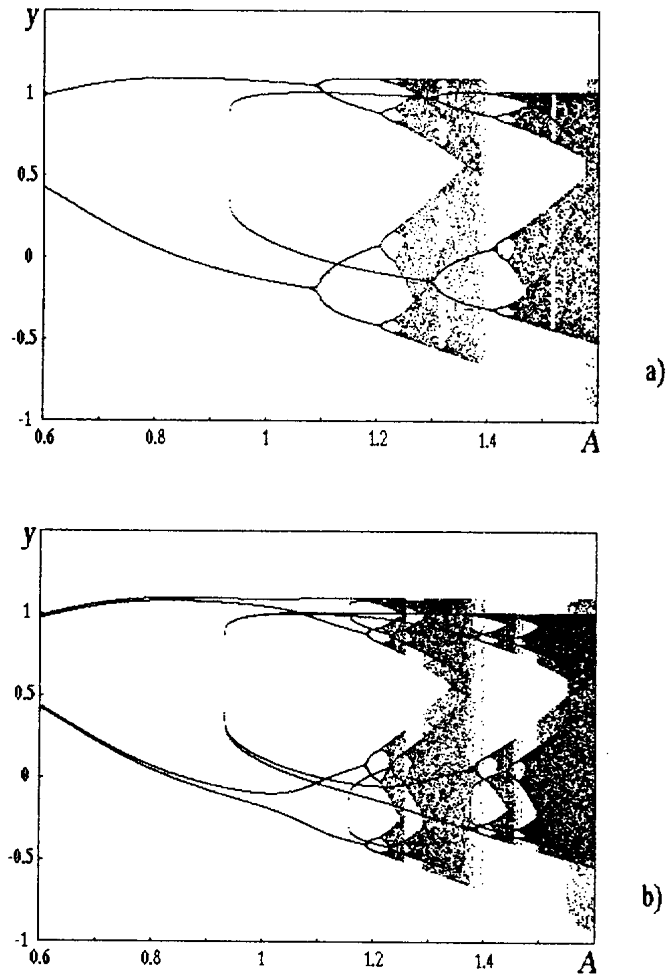

Fig. 2. Bifurcation trees for the second subsystem obtained by iteration of the initial point ensemble: (a) $\lambda = 0.85$, $B = -0.1$ (period-2 cycle in the first subsystem); (b) $\lambda = 1.28684$, $B = -0.1$ (period-4 cycle in the first subsystem).

it is just a period of external forcing produced by the first subsystem.

Let us continue further increasing λ. After the second period-doubling bifurcation in the first subsystem, a new cusp point appears on each of the above two sheets; both cusps, ${}^2C_{2^2(0)}$ and ${}^2C_{2^2(1)}$, are projected to the same point of the (A, B)-plane, namely, $(\lambda_1, 0)$. Accordingly, each of the two sheets is found to be a foliated surface [see Fig. 1(b)]. The edges of the newly formed sheets are the fold lines ${}^2f_{2^2(00)}$, ${}^2f_{2^2(01)}$ and ${}^2f_{2^2(10)}$, ${}^2f_{2^2(11)}$, respectively. Figure 2(b) shows a bifurcation diagram of the second subsystem for $\lambda = 1.28684$ and $B = -0.1$. Here one can see four overlapping "trees" relating to four sheets of Fig. 1(b). Now the period-doubling cascade in the second subsystem begins with the period-4 cycle.

The process of new cusp formation and sheet branching continues *ad infinitum* while λ goes to the critical value $\lambda_c = 1.401155\ldots$ After the nth doubling in the first subsystem, we find that the start of the period-doubling cascade in the second subsystem is given by the period-2^n cycle. The foliated (A, B)-surface have 2^n relevant sheets, $2^n - 1$ cusp points ${}^rC_{2^k(i_1 i_2 \cdots i_k)}$ and pairs of fold lines ${}^rf_{2^k(i_1 i_2 \cdots i_k 0)}$, ${}^rf_{2^k(i_1 i_2 \cdots i_k 1)}$, where $k = 1, \ldots, n$, $0 < r \leqslant k$, $i_1, i_2, \ldots, i_k = 0, 1$. Here 2^k defines a cycle period in the first subsystem, r the index of the cycle at first given rise to this cusp or fold. A sequence $i_1, i_2, \ldots, i_k$ specifies a sheet in which cusps and folds lie. Note, that all ${}^rC_{2^k(\ldots)}$ points have the (A, B) coordinates independent of r and i, namely, $(\lambda_k, 0)$.

The cusp point sequence C_{2^k} converges to the point $(\lambda_c, \lambda_c, 0)$ in the parameter space (λ, A, B). This point will be further referred to as a *double Feigenbaum's point DF*. The above process of multistable state reproduction is associated with the *DF* point just like the period-doubling cascade is associated with the Feigenbaum critical point of an individual subsystem.

We conclude that the above considered foliated surfaces, rather than simply the (A, B)-plane, give suitable "canvases" for graphical presentation of dynamics dependent on A, B with fixed λ. For convenience, we shall denote the sheets by $S0$, $S1$, $S2, \ldots$ The symbol SM is used for the sheet, if the subsystems will oscillate with a shift of M time steps ($x_n = y_{n+M}$), when we come along the sheet to the point $B = 0$, $\lambda = A$. We can examine the sheets consequently and then recognize how are they joined to form a complete foliated surface. The S_0 sheet may be called the *in-phase sheet.* It is just the facing side of the surfaces in Fig. 1. It is found that the attractors of this sheet are realized if we increase A from zero, λ and $B > 0$ being fixed. To come to any another sheet, one must move along a definite "slalom-like" path round the cusp points C_{2^k}.

3. Critical Phenomena at the In-Phase Sheet

Now we turn to the basic subject of our analysis and consider different types of criticality exhibited by the system at the onset of chaos. All the critical situations may be found at the in-phase sheet. Therefore, it is just the sheet which we shall study in detail.

Let us return to the case when the stable period-2 cycle is realized in the first subsystem. Figure 3 shows the regions of various dynamical behaviors at two sheets of the (A, B)-surface for $\lambda = 0.85$. They form a distinctive configuration called the *crossroad area* by Carcasses *et al.* [1991]. Lines of the period-doubling bifurcations D_{2^k} are accumulated at the Feigenbaum critical lines F, which are the borders of chaos. The region of period-4 cycle stability contains a new cusp point ${}^1C_{2^2(0)}$ and fold lines ${}^1f^{(1)}_{2^2(0)}$ and ${}^1f^{(1)}_{2^2(0)}$. Here the in-phase sheet branches and becomes the foliated surface itself. As a result, the crossroad area is formed. In the region of sheet overlapping two attractors coexist, being two different stable period-4 cycles near the cusp point. If we look at the parameter plane structure with better resolution (see the magnified fragment in Fig. 3), a lot of the crossroad areas become visible, containing cusps and folds based on period 8, 16, 32... cycles. Each crossroad area gives rise to two new crossroad areas, so the number of them increases as 2^n with the order of the considered period-2^n cycles. To generalize the above identification symbolism for this new set of cusps and folds, we provide the right superscripts, being a coding sequence of 0 and 1 (see Fig. 3, where the first level of the construction is shown).

Let us examine carefully the behavior of the period-doubling lines D_{2^i}. We start from $B = 0$ and increase the coupling. At moderate B values, the D_{2^i} lines lie on the same sheet and accumulate to the limit line F. Further, the lines D_{2^i} go round different cusp points and continue therefore on different sheets. It means that the Feigenbaum critical line breaks at some point. This point is an accumulation point of certain cusp point sequence

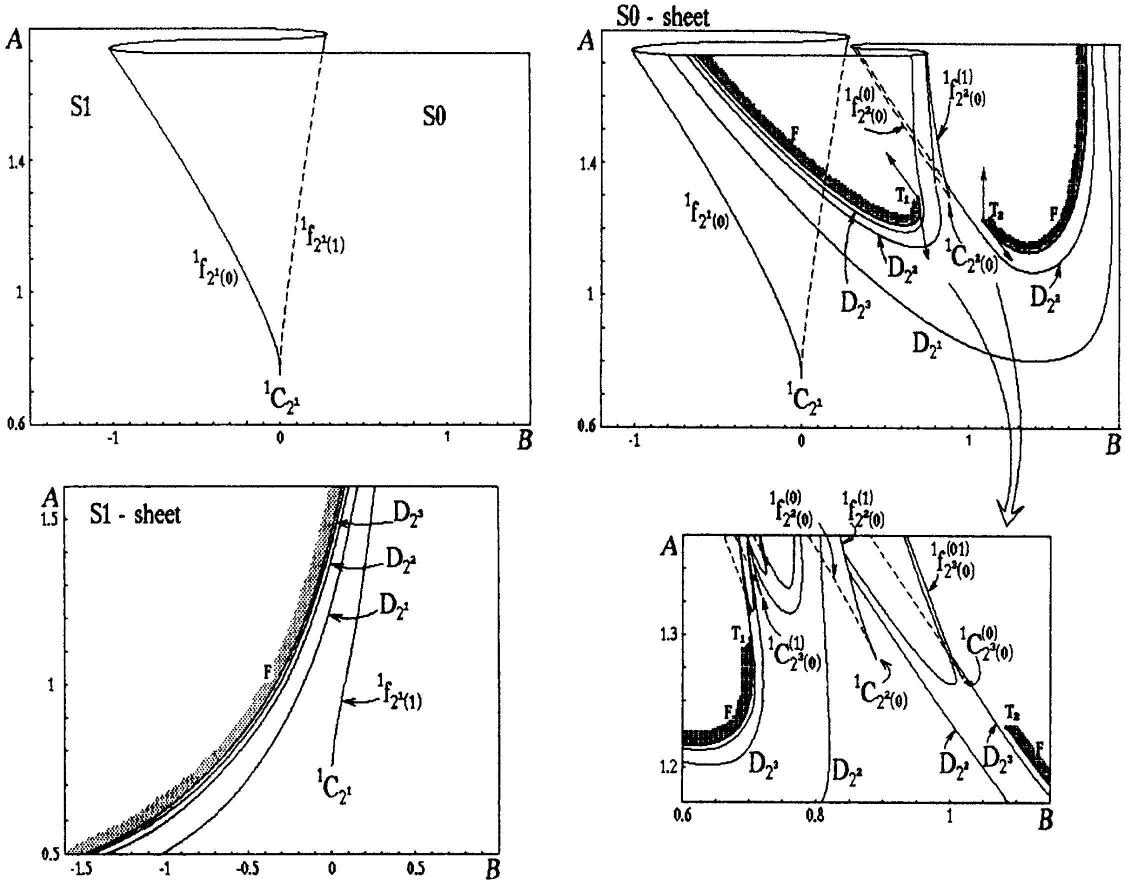

Fig. 3. Regions of different dynamical regimes in the (A, B) parameter plane for $\lambda = 0.85$. Cusp points C, fold lines f, period-doubling lines D, tricritical points T and Feigenbaum critical lines F are shown, the last lines are marked by shading from the chaotic region side. The arrows show the eigendirections for the tricritical points.

Table 1. The cusp point sequence, converging to the T1 point, $\lambda = 0.85$.

N	A_N	B_N	y_N	$\frac{A_\infty - A_N}{A_\infty - A_{2N}}$	y_N/y_{2N}
4	1.283293	0.891207	0.113696		
8	1.330607	0.716126	−0.070483	−0.466	−1.613
16	1.314539	0.701132	0.043617	1.992	−1.616
32	1.304689	0.701732	−0.026291	2.551	−1.659
64	1.300657	0.702751	0.015662	2.740	−1.679
128	1.299164	0.703224	−0.009287	2.809	−1.686
256	1.298629	0.703407	0.005498	2.845	−1.689
∞	1.298339	0.703509		2.85713	−1.69030

(see Table 1) and is called *tricritical.*

Tricritical points were introduced into consideration by Chang *et al.* [1981] when studying a one-dimensional two-parameter quartic map. Our problem may be transformed exactly into this form in the considered case of period-2 oscillations in the first subsystem. Iterating twice the Eqs. (1), we obtain a map describing the evolution of the second subsystem over two time steps:

$$y_{n+2} = [1 - Bx_1^2 - A(1 - Bx_0^2)] + 2(1 - Bx_0^2)A^2y_n^2 - A^3y_n^4\,,$$

where $x_{0,1} = [1/2 \pm (\lambda - 3/4)^{1/2}]/\lambda$ are the elements of the period-2 cycle in the first subsystem. Variable change $y \to y[1 - Bx_1^2 - A(1 - Bx_0^2)^2]$ leads to the equation:

$$y_{n+2} = 1 + ay_n^2 + by_n^4\,, \tag{2}$$

where

$$\begin{aligned} a &= 2(1 - Bx_0^2)(1 - Bx_1^2 - A(1 - Bx_0^2)^2)A^2\,, \\ b &= -[A(1 - Bx_1^2 - A(1 - Bx_0^2)^2)]^3\,. \end{aligned} \tag{3}$$

Note, that the map (2) is a particular case of the map studied by Carcasses *et al.* [1991] who found crossroad areas.

Two tricritical points given by Chang *et al.* [1981] have the coordinates $(-2.81403, 1.40701)$ and $(0, -1.59490)$ in the (a, b) plane. They can be mapped into the (A, B) plane by solving Eqs. (3); they are $T_1(1.29834, 0.70351)$ and $T_2(1.23075,$

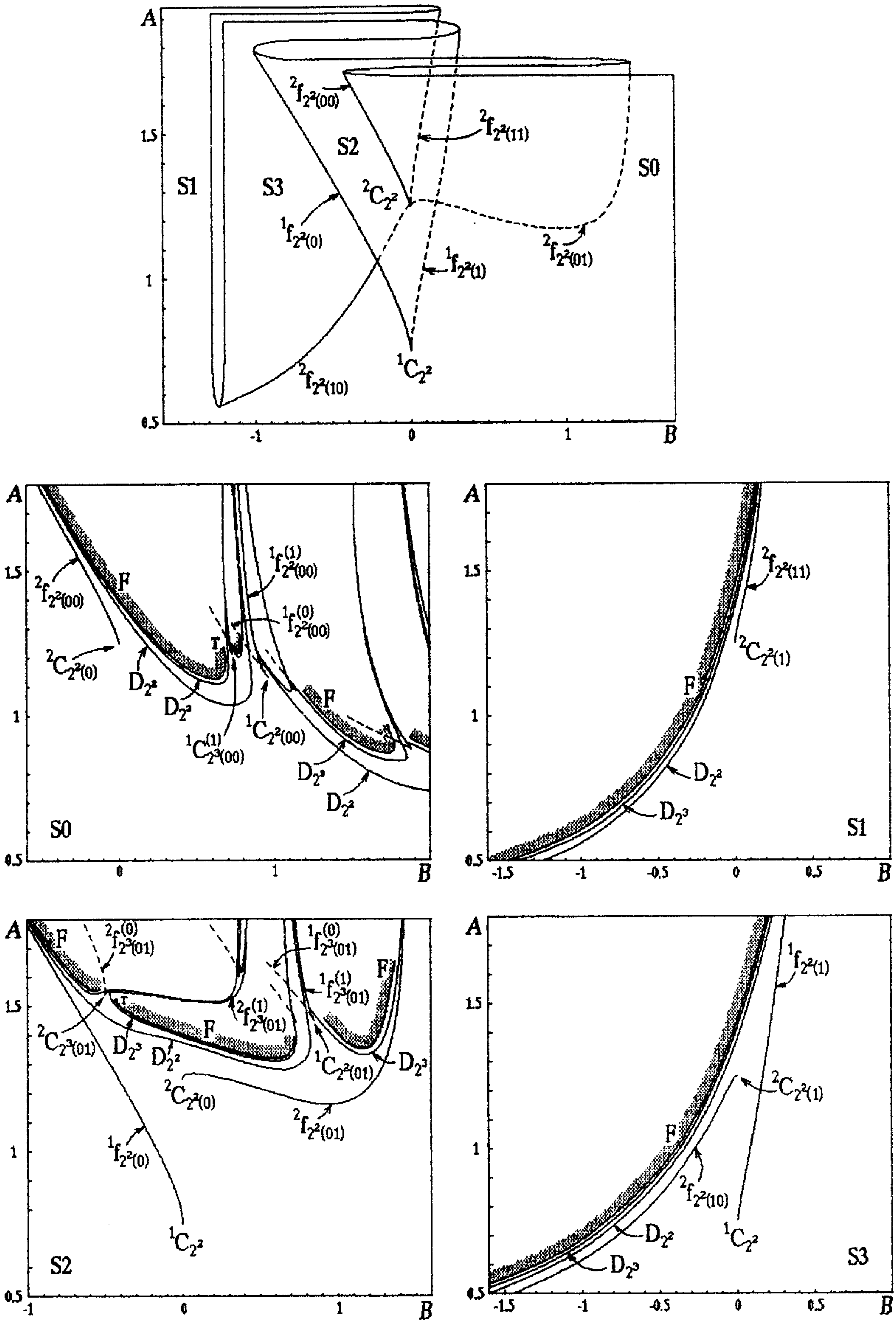

Fig. 4. Regions of different dynamical regimes in the parameter plane of the second subsystem at $\lambda = 1.28684$ (period-4 cycle in the first subsystem).

1.08446) (see Fig. 3). In fact, an infinite set of tricritical points exist, each pair associating with an element of the crossroad area hierarchy (see Kapral & Fraser [1984]).

According to the results of the RG analysis [Chang *et al.*, 1981; Kapral & Fraser, 1984], two-parameter scaling takes place in the vicinity of each of the tricritical points: the topography of the parameter plane is reproduced with scale change along suitable coordinate axes by factors of $\delta_{T1} = 7.28469$ and $\delta_{T2} = 2.85713$. The corresponding axis directions (*eigendirections*) in the (A, B)-plane are shown by arrows. The first is along the line at which the second subsystem dynamics over two time steps obeys a map with quartic extremum, after some proper dynamical variable change, and the second is along the Feigenbaum's critical line. The cusp point sequence converges geometrically with exponent δ_{T2} to the tricritical point along the last eigendirection too, but from the opposite side (Table 1).

It should be noted here, that the term "tricritical" was introduced by analogy with the phase transitions theory: the point is called tricritical if there exist phase transitions of both the first and the second orders in it's arbitrarily small vicinity [Stanley *et al.*, 1980]. The first order transitions are associated with tangent bifurcations, while the second order ones with the onset of chaos via the period-doubling cascade.

Let us now choose the parameter of the first subsystem in such a way that it would demonstrate a period-4 cycle. Taking $\lambda = 1.28684$, we present the dynamical behavior topography of the foliated (A, B)-surface in Fig. 4. Again we see the period-doubling lines D, fold lines f and cusp points C. Crossroad areas are found in regions of period-4, 8, ... cycles. As before, two types of criticality are presented, the Feigenbaum lines F and tricritical points T.

Note that searching for tricritical points becomes more difficult task than in the model map (2). We used an alternative algorithm based on an universal property of the tricriticality. A dynamical system has all unstable period-2^k cycles at the tricritical point, their multipliers converge to the universal value $\mu_c = -2.05094004903$ at $k \to \infty$ limit. Let us find the points in the parameter plane at which the multipliers of period-2^k and period-2^{k+1} cycles would be both equal to μ_c. The tricritical point is obtained in the $k \to \infty$ limit and convergence is very good.

In the same way, one can consider the situations of the first system oscillations of period-8, 16, and so on. If we have a period-2^k cycle in the first subsystem, the first period-doubling bifurcation in the second subsystem is accompanied by the appearance of the period-2^{k+1} cycle, and the cusps based on this cycle give rise to cascades of crossroad area and cusp formation.

Let us consider now the case of critical parameter value $\lambda_c = 1.401155$ in the first subsystem. In this case new types of critical behavior appear in the second subsystem. One of them, called *bicritical*, was considered by Bezruchko *et al.* [1986, 1990] and Kuznetsov *et al.* [1991]. A bicritical line in the (A, B) plane may be revealed by retracing the evolution of Feigenbaum's line in the second subsystem when $\lambda \to \lambda_c$. In Fig. 5 the Feigenbaum's line is shown in the (A, B)-plane for λ values corresponding to period doubling bifurcation of period 2, 4, 8, 16 cycles in the first subsystem. It may be seen that when the parameter λ value comes close to the critical value, the lines in the (A, B) plane accumulate at the bicritical line B. The tricritical points, in turn, accumulate at some point BT. The DF point is situated at the opposite end of the bicritical line.

To understand the scaling properties of bicriticality, we may consider any transversal surface crossing the bicritical line in the three-dimensional parameter space (λ, A, B). Then, the structure of regions of different dynamical behavior at this surface is reproduced under the scale change by factors of $\delta_{B1} = \delta_F = 4.669201$ and $\delta_{B2} = 2.392724$. The first eigendirection of the two-parameter scaling coincides with the λ-axis direction, and the second one with that of the A-axis. The bicritical point corresponds to the onset of *hyperchaos* (chaotic behavior which is characterized by two positive Lyapunov exponents). Also the term "bicritical" was adopted from the phase transition theory: two Feigenbaum critical lines meet here. Usually, this term means that the lines of the second-order transition lines meet.

Table 2. The tricritical point sequence convergence to the multicritical point BT.

λ	A_N	B_N	$\frac{A_\infty - A_N}{A_\infty - A_{2N}}$
1.25	1.279894	0.690210	
1.368099	1.173074	0.721136	1.998
1.394046	1.108720	0.784254	2.506
1.399631	1.080086	0.817598	3.034
1.400829	1.071161	0.828522	2.729
1.401155	1.066	0.83505	2.654

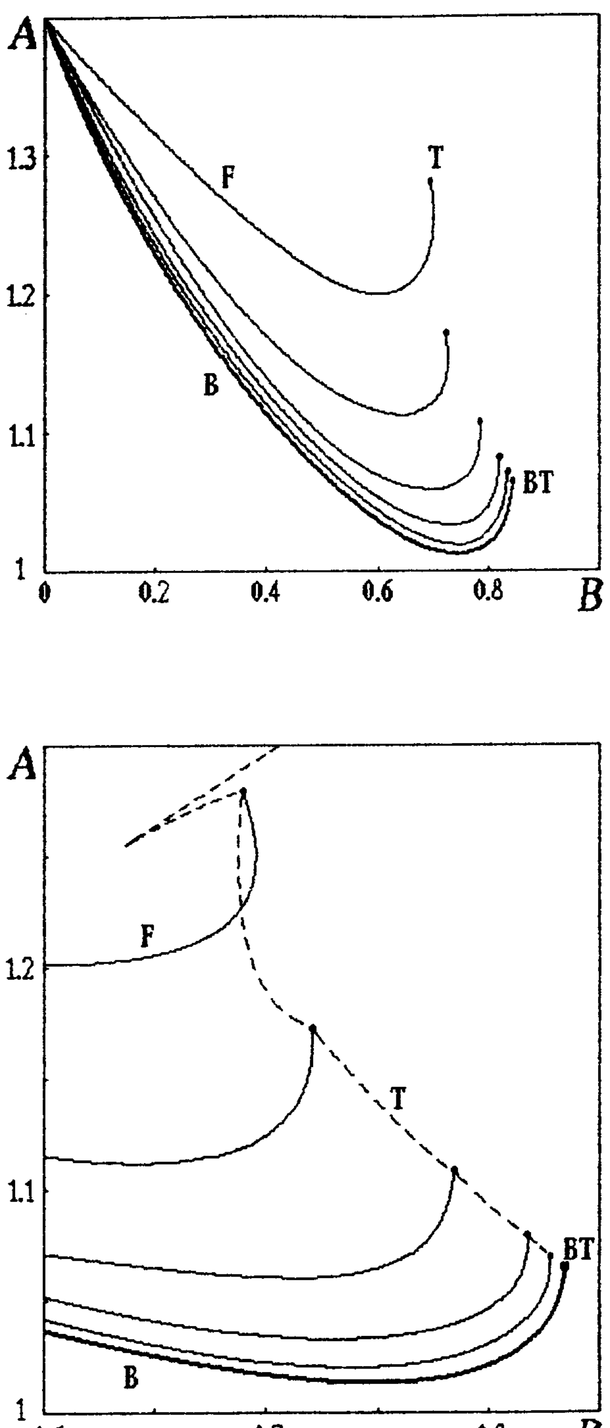

Fig. 5. Convergence of Feigenbaum's critical lines F and their end points — tricritical points T — to the bicritical line B and multicritical point BT. Critical lines are shown for parameter values $\lambda = 1.25$; 1.368099; 1.394046; 1.399631; and 1.400829.

The BT and DF points correspond to situations of codimension-3 in our map, and three-parameter scaling takes place in their vicinities.

To locate the BT point in the (λ, A, B)-parameter space with sufficient accuracy, we extrapolate the bicritical line and a tricritical line projection onto the $\lambda = \lambda_c$ plane up to the point of their intersection. This is just evaluation for the BT point coordinates at the in-phase sheet: (1.401155, 1.06620, 0.83505). Universal constants characterizing the parameter space scaling properties near the BT point are $\delta_{BT1} = \delta_F$, $\delta_{BT2} = 2.654654$ and $\delta_{BT3} = 1.541720$. The eigendirection related to the δ_{BT1} factor is along the λ-axis, tricritical point sequence accumulates along the second eigendirection and is characterized by the convergence rate δ_{BT2}. At last, the third eigendirection is the direction of bicritical line flowing out of the BT point.

The DF point correspond simply to Feigenbaum criticality in two uncoupled subsystems. So, the first two scaling factors along the λ and A axes of the parameter space are $\delta_{DF1} = \delta_{DF12} = \delta_F$. The third scaling factor is $\delta_{DF3} = 2$. The third eigendirection at the in-phase sheet is given by the line $A + B = \lambda_c$.

The above mentioned universal scaling factors intrinsic to different types of critical behavior were really obtained using the RG analysis. This approach is well known for Feigenbaum criticality and tricriticality [Feigenbaum, 1978, 1979; Chang *et al.*, 1981; Kapral & Fraser, 1984]. Therefore, in the next section, we consider the RG approach covering the critical situations B, BT and DF.

4. Renormalization Group Analysis

Denoting right-hand functions in Eqs. (1) by $f_0(x)$ and $g_0(x, y)$, one may obtain a map describing the evolution over two time steps for dynamical variables x and y rescaled by some factors a and b in the following form

$$x_{n+2} = ag_0(g_0(x_n/a))\,,$$
$$y_{n+2} = bf_0(g_0(x_n/a,\ f_0(x_n/a,\ y_n/b))\,.$$

Let us denote the new right-hand functions by $g_1(x)$ and $f_1(x, y)$. The recurrent RG equation is obtained by multiple repetition of the above procedure:

$$g_{m+1}(x) = ag_m(g_m(x/a))\,,$$
$$f_{m+1}(x) = bf_m(g_m(x/a),\ f_m(x/a,\ y/b))\,.$$

Critical behavior corresponds to a fixed point of this functional map [Kuznetsov *et al.*, 1991]:

$$g(x) = ag(g(x/a))\,,$$
$$f(x, y) = bf(g(x/a),\ f(x/a,\ y/b))\,. \tag{4}$$

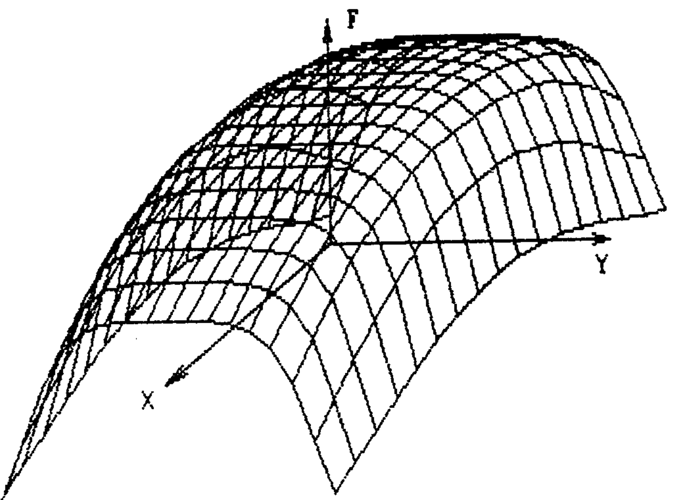

Fig. 6. Universal function, describing the dynamics at the multicritical point BT.

To solve these equations means to find the functions g, f and the factors a, b. Note that under the normalization conditions $g(0) = 1$, $f(0, 0) = 1$, we have $a = (g(1))^{-1}$ and $b = (f(1, 1))^{-1}$.

The first of Eqs. (4) describes the behavior of the first subsystem and is independent of the second one. The well-known Feigenbaum's function $g(x)$ gives a solution to this equation with factor $a = a_F = -2,502907$.

The second of Eqs. (4) has several solutions which are of interest to us now. They are responsible for different critical situations in the second subsystem. The simplest of them is $f(x, y) = g(y)$ with $b = a_F$. It corresponds to the DF point. To find the other solutions, we must solve the system (4) numerically. The function $f_B(x, y)$ for bicritical behavior may be searched for in the form of a polynomial expansion involving powers of x^2 and y^2, and that for the BT case, $f_{BT}(x, y)$, involving powers of x^2 and y^4. A polynomial approximation for the function $f_B(x, y)$ was found by Kuznetsov *et al.* [1991], the scaling factor being $b = a_B = -1,505318159$. A polynomial approximation for the $f_{BT}(x, y)$ function is presented in the Appendix. The plot of the function is shown in Fig. 6. The scaling factor is $b = a_{BT} = -1,241661$.

Note that, to search for the BT points, one could use an algorithm similar to that suggested in Sec. 3 for the tricriticality, using the universal critical multiplier value $\mu_c = -1.398015$. However, convergence of this algorithm is rather weak.

The next step of the RG analysis consists in studying the evolution of small perturbations of the fixed point functions $g(x)$, $f(x, y)$ under the successively applied RG transformation. The first unstable direction of the RG fixed point corresponds to a perturbation making the first subsystem go out from the critical point. It is represented by the known Feigenbaum's function $h_F(x)$ and the universal number δ_F [Feigenbaum, 1978, 1979]. The other unstable directions correspond to the perturbations of the second subsystem only. Substituting $f_m(x, y) = f(x, y) + \delta^m H(x, y)$, one obtains the following eigenvalue problem:

$$\begin{aligned}\delta H(x, y) = b[&f'_y(g(x/a), f(x/a, y/b))H(x/a, y/b) \\ &+ H(g(x/a), f(x/a, y/b))]\,. \end{aligned} \tag{5}$$

Taking the above functions f_{DF}, f_D, and f_{BT} as $f(x, y)$ in the Eqs. (5), one may obtain the other relevant eigenvalues δ and the eigenfunctions $H(x, y)$ for all the types of criticality.

For the DF point, we substitute $f(x, y) = g(y)$ and $b = a_F$ into the Eqs. (5), and come to the equation [Kuznetsov, 1985; Aranson *et al.*, 1988; Kook *et al.*, 1991]

$$\begin{aligned}\delta H(x, y) = a_F[&g'(g(y/a_F))H(x/a_F, y/a) \\ &+ H(g(x/a_F), g(x/a_F))]\,. \end{aligned}$$

Since the model (1) includes even powers of x and y only, we restrict ourselves to even solutions of Eq. (5). The first solution is rather trivial: $H(x, y) = h_F(y)$, $\delta_{DF} = \delta_F$. Another solution has the eigenvalue $\delta_{BT3} = 2$, and the following polynomial approximation may be found for the eigenfunction:

$$\begin{aligned}H_{DF3}(x, y) = &-1.0586844y^2 + 0.0547721y^4 \\ &+ 0.004464y^6 - 0.0005518y^8 \\ &+ 1.0586824x^2 + 0.0357628x^2y^2 \\ &+ 0.003456x^2y^4 - 0.0001064x^2y^6 \\ &- 0.0905175x^4 + 0.0351256x^4y^2 \\ &- 0.0010823x^4y^4 - 0.0430980x^6 \\ &- 0.0050244x^6y^2 + 0.0001013x^6y^4 \\ &+ 0.0068283x^8 + 0.0001738x^8y^2 \\ &- 0.0003016x^{10}\,. \end{aligned}$$

It may be shown that there are no other relevant solutions in the even subspace. Thus, the DF point has three relevant eigenvalues $\delta_{DF1} = \delta_F$, $\delta_{DF2} = \delta_F$, $\delta_{DF3} = 2$, and codimension-3.

For bicriticality B it has been found that the Eq. (5) has the only greater-than-unity eigenvalue

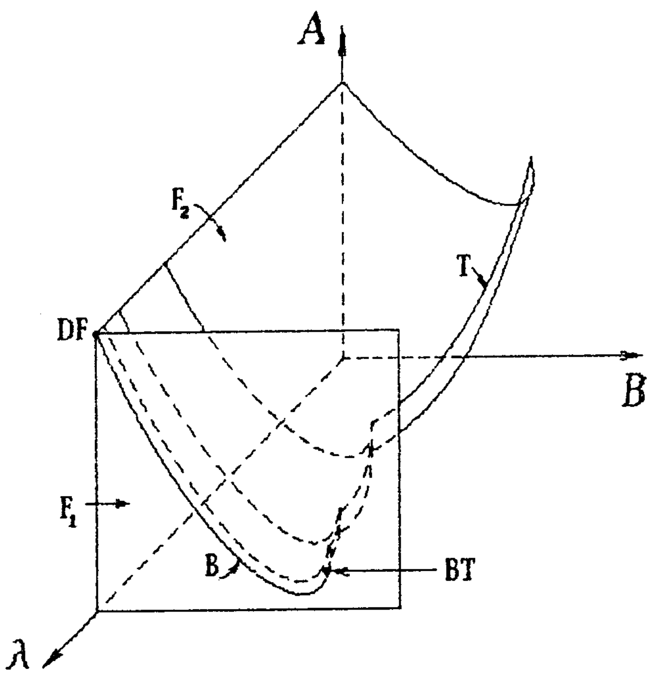

Fig. 7. Geometry of different types of critical dynamics in the three-dimensional parameter space: $F1$ and $F2$ — Feigenbaum's critical surfaces, T — tricritical lines, B — bicritical lines, DF — double Feigenbaum's point, BT — multicritical point.

$\delta_{B2} = 2.39272443$ [Kuznetsov *et al.*, 1991]. Adding the eigenvalue $\delta_{B1} = \delta_F$, we conclude that the codimension is two.

For the BT point two relevant solutions of Eq. (5) exist. The corresponding eigenvalues are $\delta_{BT2} = 2.654654$ and $\delta_{BT3} = 1.54172$. Accounting for $\delta_{BT1} = \delta_F$, we find that the resultant codimension of the BT point is three.

5. Hierarchy of Critical Dynamics: General Discussion

Let us sum up the results. In the (λ, A, B) space there are two Feigenbaum's critical surfaces $F1$ and $F2$ (Fig. 7). The first one is a plane $\lambda = \lambda_c$, and the second one is a surface of complex form, the latter being a boundary of regions with chaotic behavior in the second subsystem. These surfaces intersect along the bicritical line B. The surface $F2$ has a boundary — a line of tricritical points T. The tricritical and bicritical lines meet and terminate at the multicritical BT point. The second end point of the bicritical line is the DF point. Note, that

Table 3. Quantitative characteristics for the hierarchy of critical dynamics in the case of unidirectionally coupled Feigenbaum's systems.

Critical point type	n	m	δ	α	μ	$\gamma\,dB$	æ	χ
Feigenbaum's F	1	1	4.66920	−2.50291	−1.6012	13.35	0.69	0.4498
Tricritical	2	1	7.28469	−1.69030	−2.0509	10.40	0.40	0.3491
T			2.85712					0.6603
Bicritical	2	2	4.66920	−2.50291	−1.6012	13.35	0.69	0.4498
B			2.39272	−1.50532	−1.1789	7.98	0.92	0.7945
Multicritical	3	2	4.66920	−2.50291	−1.6012	13.35	0.69	0.4498
BT			2.65465	−1.24166	−1.3980	6.85	0.82	0.7100
			1.54172					1.6012
Double	3	2	4.66920	−2.50291	−1.6012	13.35	0.69	0.4498
Feigenbaum's			4.66920	−2.50291	−1.6012	13.35	0.69	0.4498
DF			2.00000					

n — number of relevant parameters (codimension),
m — number of relevant dynamical variables,
δ — scaling factor in the parameter space,
α — scaling factor in the phase space,
μ — period-2^n cycle multiplier at the critical point,
γ, æ — factors, characterizing the spectra of oscillations at the critical point (overfall between neighboring subharmonic levels and nonuniformity of subharmonic amplitude distribution for a given level of hierarchy),
χ — critical indices for Lyapunov exponent.

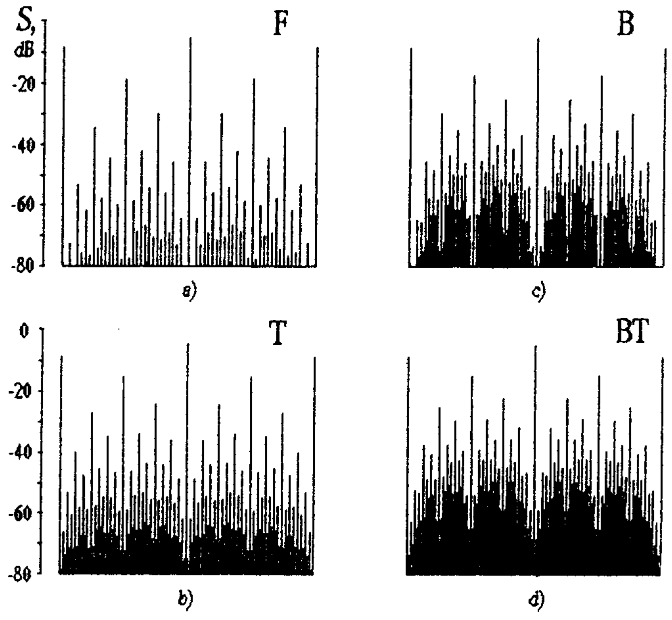

Fig. 8. Spectra of oscillations in the second subsystem for different critical situations: a) Feigenbaum's point, b) tricritical point, c) bicritical point, d) multicritical point BT.

Table 4. Multicritical BT points in the (A, B) plane.

SM	A	B
0	1.0662	0.83505
1	0.43845	−1.8155
2	1.38117	−0.26121
3	1.63509	0.18750
4	1.46430	−0.11247
5	1.59822	0.09508
6	1.37970	0.12175
7	1.26741	−0.10066
8	1.44558	−0.05397
9	1.50961	0.04807
10	1.40897	0.05524
11	1.36147	−0.03947
12	1.35843	0.06848
13	1.27107	−0.07521
14	1.40186	−0.04563
15	1.44660	0.03451

any small vicinity of the tricritical line, BT and DF points contains multistability regions and tangent bifurcation surfaces.

A definite number of quantitative characteristics is associated with each of the above-listed types of critical behavior (the number of relevant parameters and dynamical variables, scaling factors in the phase and parameter spaces, critical multipliers of period-2^k cycles and so on). Some of them are presented in Table 3.

Table 3 also contains scaling factors characterizing the spectra of oscillations in the second subsystem for different critical situations under consideration — Feigenbaum's, tricritical, bicritical and BT. The spectra themselves are presented in Fig. 8. Each of them has it's own special form and may be useful for identification of the type of critical dynamics, for example, in experiment. The spectra possess strongly pronounced hierarchical organization of subharmonic levels and may be approximated by the recurrent relation

$$\frac{\alpha^2+\beta^2 \pm 2\alpha\beta\cos(\pi\omega/2)}{4\alpha^2\beta^2}, S(\omega) \to \begin{cases} S(\omega/2)\,, & \text{sign " + "}\,, \\ S(1-\omega/2)\,, & \text{sign " − "}\,. \end{cases} \qquad (6)$$

Here α is a scaling factor for the dynamical variable y in the corresponding critical situation (see Table 4), β equals either α^2 for Feigenbaum's and bicritical points or α^4 for tricritical and BT points. The Eq. (6) is a generalization of the well-known expression for the Feigenbaum's spectrum obtained by using the same method [Huberman & Zisook, 1981; Nauenberg & Rudnik, 1981].

As can be seen from Eq. (6), a value of $\gamma = (1/4\alpha^2 + 1/4\beta^2)$ characterizes a mean overall between neighboring subharmonic levels, while a value of $æ = |2\alpha\beta/(\alpha^2+\beta^2)|$ characterizes a degree of nonuniformity of subharmonic amplitude distribution for a given level of hierarchy.

Lyapunov exponent behavior near the critical points obeys scaling relations of the next type:

$$L \to L/2\,, \quad \Lambda \to \Lambda/\delta\,, \qquad (7)$$

where Λ is a parameter value count off along some eigendirection, and δ is a scaling factor for this direction. Hence a relation may be obtained for the Lyapunov exponent envelope

$$L \simeq \Lambda^{\chi}\,, \qquad (8)$$

where the critical exponent χ is defined as $\chi = \ln 2/\ln\delta$. The values of χ are also presented in Table 3.

The structure of other SM sheets appears to be analogous to that of the in-phase sheet which was discussed in detail. The same types of critical dynamics with the same values of scaling factors are

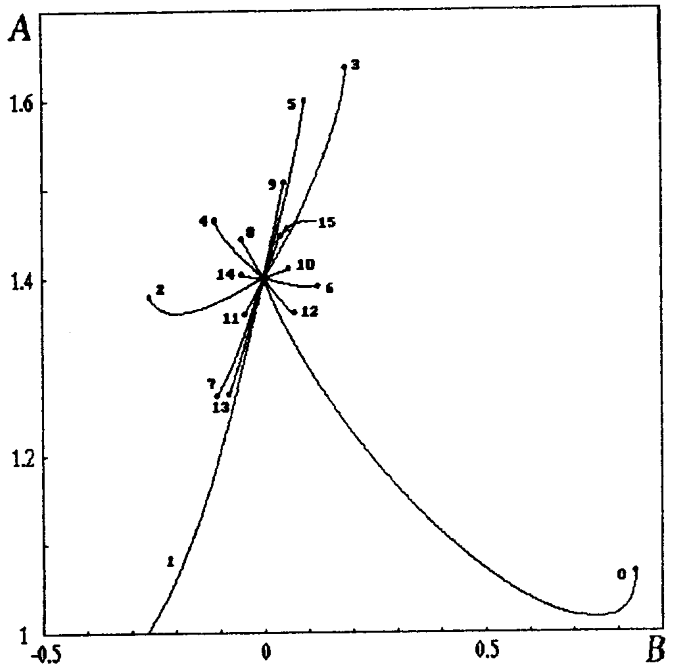

Fig. 9. Bicritical lines and multicritical points in the (A, B) plane — "bicritical star." The integers $0, \ldots 15$ are the numbers of the $S0, \ldots S15$ sheets where the corresponding lines lie.

observed. It is an illustration of critical phenomena universality. However, the location of the critical points and lines inside the sheet boundary is not universal.

Figure 9 presents a resultant global arrangement of critical phenomena for different sheets in the vicinity of the *DF* point at $\lambda = \lambda_c$: shown there are bicritical lines in the (A, B) plane up to sheet order $M = 15$. It is of a star-shaped type with a center at the *DF* point. The orders of the sheets are shown near the rays. Each ray is broken by the multicritical *BT* point. The coordinates of all the *BT* points shown are presented in Table 4. One half of rays are located in the semi-plane $B < 0$ and the second half in the semi-plane $B > 0$. Tangent bifurcations are observed near the *DF* point from the opposite side to the bicritical "ray" side in the boundaries of each sheet, and there are no known critical behavior here.

6. Conclusion

Thus, so far we have shown *the concept of "moving up codimension" to be fruitful for the theory of critical phenomena at the onset of chaos.* From this point of view, the main tasks for the theory are searching for and classifying typical critical dynamics which are dependent on relevant parameter numbers, revealing their inherent universality and scaling properties, and establishing canonical models describing each critical situation. The rules for the coexistence of critical dynamics types in the parameter space and the methods for searching and identifying them in experiment are also subjects of research.

Today, a great number of systems demonstrating the classical period-doubling scenario of transition to chaos are known. It is widespread since the Feigenbaum's criticality arises typically in systems with one control parameter ($n = 1$, see Table 3). What may be said about possible realizations of the other types of critical dynamics considered in this paper?

Tricritical behavior is typical in two-parameter families of nonlinear systems describing by bi- and multi-modal one-dimensional maps [Kapral & Fraser, 1984]. An indication of the tricritical situation is a characteristic structure of the parameter plane — the presence of cusp point and crossroad area hierarchy and the distinctive period-doubling lines topology (Fig. 3). A similar structure of the parameter plane can be seen in a number of theoretical and experimental works. A splendid specimen of this pattern referring to Chua's circuit [Komuro *et al.*, 1991] is featured on the front page of the very first issue of the *International Journal of Bifurcations and Chaos.* It is reasonable to search for tricriticality in such situations. The considered case of unidirectionally coupled systems is only one of many possible examples.

The bicritical behavior and multicritical *BT* point are, apparently, characteristic for open flow systems only, since introducing feedback destroys these types of critical dynamics. Experimental observations of bicritical behavior was done by Bezruchko *et al.* [1986]. It should be noted that critical phenomena of such type may occur in a chain of three, four and greater number of elements with unidirectional coupling, when an independent monitoring subsystem parameter is possible.

As for the *DF* point and related multistability, they are characteristic for unidirectionally as well as for mutually coupled period-doubling systems. Experimentally such multistability was observed by Bezruchko *et al.* [1990].

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Appendix

The universal function for the BT critical point:

$$
\begin{aligned}
F(x, y) = {} & 1 - 0.401489y^4 - 0.647008y^8 - 0.764338y^{12} + 0.948317y^{16} - 6.512318y^{20} + 23.444001y^{24} \\
& - 51.462786y^{28} + 76.526244y^{32} - 69.236797y^{36} + 33.523789y^{40} - 6.654459y^{44} + x^2(-0.380946 \\
& - 0.578270y^4 - 0.924782y^8 + 3.207539y^{12} - 27.026345y^{16} + 127.361506y^{20} - 364.218724y^{24} \\
& + 676.471413y^{28} - 793.030007y^{32} + 560.113033y^{36} - 217.468288y^{40} + 35.712535y^{44}) \\
& + x^4(-0.150141 - 0.266129y^4 + 0.922201y^8 - 13.913457y^{12} + 99.471585y^{16} - 387.166982y^{20} \\
& + 937.925631y^{24} - 1420.505654y^{28} + 1331.993505y^{32} - 749.454686y^{36} + 231.199927y^{40} \\
& - 29.871488y^{44}) + x^6(-0.024968 - 0.021137y^4 - 1.107687y^8 + 23.366570y^{12} - 149.729985y^{16} \\
& + 512.519022y^{20} - 998.517972y^{24} + 1136.063481y^{28} - 749.225002y^{32} + 266.722853y^{36} \\
& - 40.001237y^{40}) + x^8(-0.002388 + 0.228128y^4 - 0.111086y^8 - 13.210179y^{12} + 106.642684y^{16} \\
& - 339.857148y^{20} + 542.154904y^{24} - 460.181747y^{28} + 197.645364y^{32} - 33.358334y^{36}) \\
& + x^{10}(0.016764 - 0.176914y^4 + 1.694251y^8 - 1.908288y^{12} - 24.025646y^{16} + 86.009698y^{20} \\
& - 118.632825y^{24} + 76.052577y^{28} - 19.013144y^{32}) + x^{12}(-0.003513 + 0.069849y^4 \\
& - 0.891185y^8 + 3.015233y^{12} - 3.654160y^{16} + 1.461664y^{20}) \,.
\end{aligned}
$$