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A variety of period-doubling universality classes in multi-parameter analysis of transition to chaos

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Abstract

In multi-parameter analysis of the onset of chaos non-Feigenbaum period-doubling behavior may occur at some special paths in the parameter space. There are two possibilities: (i) the dynamics at the onset of chaos remains essentially one-dimensional, but the one-dimensional map is distorted in such a way that leaves Feigenbaum's universality class; (ii) a new mode comes to the threshold of instability and increases the effective dimension of the dynamics. We submit a list of one-dimensional and two-dimensional maps which represent distinct classes of the period-doubling universality, discuss the properties of the associated types of critical behavior at the border of chaos, and demonstrate pictures of the universal parameter space arrangement near the critical points.

Keywords: Onset of chaos; Universality; Period-doubling; Renormalization group; Scaling

1. Introduction

Description of turbulence as dynamical process in spatially extended systems attracts much attention of researchers. An important aspect of the problem is the question: how does the spatio-temporal chaos arise from simple regular regimes when we tune control parameters?

The breakthrough in understanding the onset of chaos in low-dimensional systems is connected with Feigenbaum's discovery of the period-doubling universality and the renormalization-group (RG) approach [1,2]. The simplest class of systems which exhibit the Feigenbaum type of behavior is represented by one-dimensional non-invertible maps. The same type of the period-doubling transition to chaos occurs in multi-dimensional dissipative nonlinear systems. However, as long as the Feigenbaum theory is valid for a spatially extended system, it permits to understand only the onset of regimes with restricted complexity of the spatial patterns.

Let us have some multi-dimensional dissipative system with several control parameters. The Feigenbaum behavior at the onset of chaos is a phenomenon of codimension 1. For example, if we have a system with three control parameters, the period-doubling bifurcations will occur at some two-dimensional surfaces in three-dimensional

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parameter space, and the sequence of these surfaces will converge to the border of chaos – the Feigenbaum critical surface.

How could we discover distinct types of dynamics at the onset of chaos? It is possible that walking along the border of chaos in the parameter space, we meet non-Feigenbaum behavior at some critical curves (phenomena of codimension 2), or at critical points (phenomena of codimension 3). We believe that the types of behavior revealed in such a way allow the RG analysis and exhibit specific properties of quantitative universality and scaling. In such a case we speak about a definite type of critical behavior (or a type of criticality).

There are two possibilities for arising non-Feigenbaum critical behavior associated with period-doubling. First, the dynamics at the onset of chaos may remain essentially one-dimensional, but the associated one-dimensional map is distorted in such a way that leaves the Feigenbaum universality class. Second, a new mode may come to the threshold of instability and increase the effective dimension of the dynamics. Certainly, the second alternative seems more interesting for understanding the spatio-temporal chaos, but one must take into account the first alternative as well to have more or less complete classification scheme for a variety of the period-doubling phenomena.

We present here a list of some types of the critical behavior that may occur in multi-parameter analysis of the onset of chaos. For each type we give all relevant numerical data and a simple example of one- or two-dimensional map being a representative of the universality class.

In Section 2 we discuss the period-doubling critical situations which are not connected with increase of the phase space dimension; they can be found and studied in one-dimensional maps. The next two sections are devoted to types of criticality that arise due to involving an additional dimension of the phase space, hence, two-dimensional maps are invoked as model systems. In Section 3 we consider a special case when the system may be decomposed into two elements with uni-directional coupling. In Section 4 we discuss other types of the period-doubling criticality intrinsic to two-dimensional maps.

2. Types of critical behavior intrinsic to one-dimensional maps

Each type of criticality discussed in this section is associated with a solution of the Feigenbaum–Cvitanović RG equation [1,2]:

$$g(x) = \alpha g\left(g\left(x/\alpha\right)\right), \tag{1}$$

where α is a universal number (scaling factor) specific for a given type of criticality. To solve Eq. (1) means to find both a function $g(x)$ (usually numerically, as a polynomial expansion) and a constant α. It is commonly used to speak about $g(x)$ as a fixed point of the RG equation.

In fact, the function $g(x)$ represents the asymptotic form of the properly normalized evolution operator for 2^k iterations of original one-dimensional map precisely at the critical point. The existence of the limit for $k \to \infty$ is ensured by rescaling of the dynamical variable as $x \sim \alpha^{-k}$.

The next step consists in analysis of small perturbations to the fixed point of the RG equation. Suppose, in the original map we produce a small parameter shift from the critical point. Now the evolution operator defined over 2^k iterations will contain some perturbation to the function $g(x)$, and it may be studied using linearized RG equation. Thus, we come to the following eigenproblem [1,2]:

$$\nu u(x) = \alpha[g'(g(x/a))u(x/a) + u(g(x/a))]. \tag{2}$$

The solutions of Eq. (2) give modes of perturbation for the fixed point $g(x)$, and only the modes with $|\nu| > 1$ will be relevant in asymptotics of large k. A number n of such eigenvalues (with exception of those associated with infinitesimal variable changes) defines *the codimension* of the criticality type; this is the minimal number of control parameters, or the minimal dimension of the parameter space, for which the critical situation may appear at some point. Indeed, the requirement that the coefficients at n relevant modes vanish, imposes precisely n conditions on parameters of the original map. If they are satisfied, we hit the critical point.

In appropriate coordinate system (*"scaling coordinates"*) the n-dimensional parameter space will have locally a universal topography, specific for the given type of criticality.

To define the scaling coordinates, we require the following condition: a shift in the parameter space from the critical point along each coordinate axis must generate a perturbation containing only one relevant eigenfunction in the solution of the linearized RG equation. The relevant eigenvalues $\nu = \delta_1, \ldots, \delta_n$ are then *the scaling factors*: if we use these values as factors of enlargement along the respective coordinate axes, we will observe similar patterns of the parameter space topography in smaller and smaller neighborhoods of the critical point.

For codimensions higher than 1, the problem to find explicitly the scaling coordinates may not be trivial, and the proper form of the parameter change depends on concrete relations between the relevant eigenvalues (see discussion in [3–5]).

Suppose we have a critical point of codimension 2, and $\delta_1 > \delta_2^k > 1$ for $k = 1, \ldots, K$, but $\delta_2^{K+1} > \delta_1$. Then, the expressions for two control parameters of the original map via the scaling coordinates (C_1, C_2) must be written in a form containing the terms $C_1, C_2, C_2^2, \ldots, C_2^K$. Only in the case $K = 1$ it is sufficient to use a linear parameter change. For codimension 3, the terms like $C_2^{k_1} C_3^{k_2}$ are to be accounted while $\delta_2^{k_1} \delta_3^{k_2} < \delta_1$. (As examples, see below the relations written out for the scaling coordinates of model maps.) It is worth noting that the same considerations remain valid for two-dimensional maps, and it will be used in the next sections.

An important number associated with any type of criticality is *a universal multiplier*. It can be calculated with high precision from the results of the RG analysis, as the derivative of the function $g(x)$ at its fixed point $x_\star = g(x_\star)$: $\mu_\star = g'(x_\star)$. In concrete models this is the value of a multiplier for orbits of period 2^k at the critical point in asymptotics of large k. This fact may be useful for accurate numerical calculations of the critical point location in the parameter space. For this, we should find such values of control parameters at which the multipliers are equal to $\mu_\star$ for an appropriate number of orbits of sufficiently long periods 2^k.

2.1. Feigenbaum criticality (type F)

This is the commonly known type of period-doubling behavior associated with the fixed point solution of the RG equation (1) [2]:

$$\begin{aligned} g(x) = 1 &- 1.527633x^2 + 0.104815x^4 + 0.026706x^6 - 0.003527x^8 \\ &+ 0.000082x^{10} + 0.000025x^{12} - 0.000003x^{14}, \end{aligned} \tag{3}$$

with $\alpha = -2.502907876$. (Here and further all presented solutions of the RG equations are normalized to unity at the origin. This is the most often used convention and the only arbitrariness in definition of the solutions.)

The eigenproblem (2) has one relevant eigenvalue $\nu = \delta = 4.669201609$, thus the codimension is equal to 1: $\mathrm{CoDim_F} = 1$.

A classic example of the Feigenbaum criticality is given by the logistic map

$$x_{n+1} = 1 - \lambda x_n^2, \tag{4}$$

the critical point is located at $\lambda_c = 1.401155189092$ [6,7].

At the critical point the map has an infinite denumerable set of unstable cycles of period 2^k. Asymptotically, for $k \to \infty$, all these cycles have the same universal multiplier $\mu = -1.6011913$. Attractor at the critical point is a fractal set with Hausdorff dimension $D_0 = 0.53840\ldots$ [8]. Locally, near the extremum $x = 0$, the structure of the attractor is reproduced under rescaling of x by the factor α. (Note that for all other types of criticality considered in this section, the critical attractors are fractal sets analogous to the Feigenbaum attractor, but with distinct quantitative characteristics.)

The one-dimensional parameter space topography near the Feigenbaum critical point contains a sequence of domains of stable regular dynamics with periods 2^k in subcritical region $\lambda < \lambda_c$, and complicated set of domains of periodic and chaotic dynamics in supercritical region $\lambda > \lambda_c$ [9–11]. This structure has a property of scaling: in small scales the parameter space arrangement reproduces itself under magnification by the factor δ [1].

2.2. The tricritical behavior (type T)

It is associated with a solution of the RG equation represented by expansion in powers of x^4:

$$\begin{aligned} g(x) = 1 &- 1.834108x^4 + 0.012962x^8 + 0.311902x^{12} - 0.062015x^{16} \\ &- 0.037539x^{20} + 0.017647x^{24} + 0.001938x^{28} - 0.002820x^{32} \\ &+ 0.000115x^{36} + 0.000399x^{40} - 0.000024x^{44} - 0.000122x^{48} \\ &+ 0.000070x^{52} - 0.000018x^{56} + 0.000002x^{60} \end{aligned} \tag{5}$$

with $\alpha = -1.6903029714$ [12,13].

The eigenproblem (2) has three relevant eigenvalues $\delta = 7.284686217$, α^2, α^3 [12,14–16]. It implies that the codimension is equal to 3: $\mathrm{CoDim}_\mathrm{T} = 3$.

To clarify this result, let us ask how many conditions should be satisfied to find the tricritical point for one-dimensional map? First, there are two conditions to have an extremum of the fourth power: the second and third derivatives must be zero at this point. One more parameter is necessary because we are to come to the critical point by tracing the period-doubling cascade, and keeping valid the previous two conditions. So, the total number of parameters is 3, in precise accordance with the estimated codimension.

As a concrete example, let us take a general smooth one-dimensional map with terms of power from 0 to 4 in its Taylor expansion. We can eliminate the cubic term by a shift of origin, and use scale change to normalize the constant term to unity. As a result, we have the following three-parameter quartic map:

$$x_{n+1} = 1 - Ax_n^2 - Bx_n^4 - Cx_n. \tag{6}$$

If $A = 0$, and $C = 0$, it demonstrates the period-doubling cascade while increasing B. The limit is the tricritical point located at $B_c = 1.594901356229$ [12]. Nevertheless, presence of three parameters in the model map is relevant because it corresponds precisely to the codimension found from the RG analysis. Due to this, the model map allows to study the parameter space structure intrinsic to a three-parameter vicinity of the tricritical point.

At the tricritical point an infinite set of unstable period-2^k cycles is presented, and the universal asymptotic value of multiplier is $\mu = -2.05094049$. Hausdorff dimension of the tricritical attractor is $D_0 = 0.642575$.

Now we have to say that in fact the question on codimension of the tricriticality in one-dimensional maps is more subtle than what follows from the above discussion. The reason is that the fourth power of extremum may appear not in the original map, but in its iteration. In this case tricriticality in one-dimensional maps may occur as a phenomenon of codimension 2 [17]. Suppose we have a smooth one-dimensional map with two quadratic extrema and with two control parameters. In general, a curve may exist in the parameter plane where the condition is valid that one extremum is mapped precisely to another. Obviously, if we stay at this curve, the iterated map has an extremum of the fourth power. If the period-doubling cascade occurs along this curve, the limit will be the tricritical

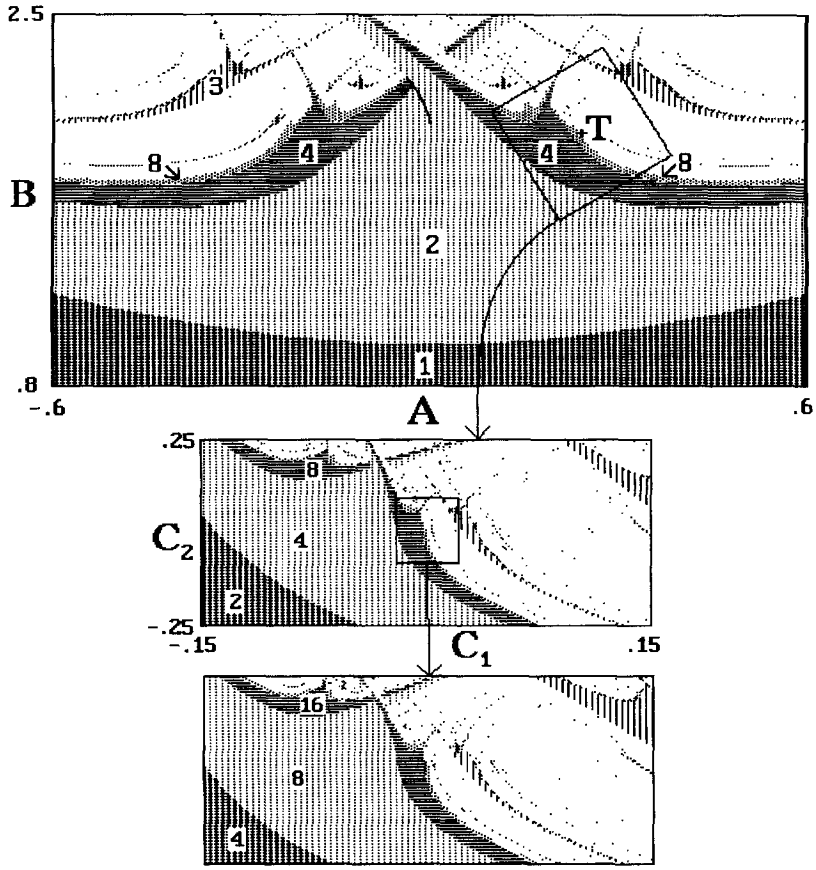

Fig. 1. Parameter plane topography for the model map (7). In this and further figures domains of different periodic regimes are shown by different shading, and periods are indicated by numbers. In the lower plots the scaling coordinates (8) are used, and the tricritical point is located in the center.

point. It may be shown that in such a case the eigenvalue α^3 is excluded (a sort of *hidden symmetry*, see [14]), and only $\delta_1 = \delta$ and $\delta_2 = \alpha^2$ are relevant. For example, the map

$$x_{n+1} = A - Bx_n + x_n^3 \tag{7}$$

has such a tricritical point at $A_c = 0.2426987573$, $B_c = 1.9513857778$.

The parameter space topography near the tricritical point of codimension 2 is illustrated in Fig. 1 for the model map (7). To show scaling properties of the picture we use coordinates (C_1, C_2) defined by the numerically fitted expressions

$$A - A_c = -0.5998610C_1 + 0.2192807C_2, \qquad B - B_c = C_1 + C_2. \tag{8}$$

In general dissipative systems which cannot be reduced to one-dimensional maps, the tricriticality of codimension 2 does not occur because the third eigenmode of the linearized RG equation inevitably enters into play. Thus, it will be necessary to have three control parameters to observe true tricriticality [14].

2.3. S-type criticality ("six-power")

The S-type criticality is associated with a solution of the RG equation represented by expansion in powers of x^6:

$$\begin{aligned} g(x) = 1 &- 1.907736x^6 - 0.332883x^{12} + 0.712702x^{18} + 0.035179x^{24} \\ &- 0.272460x^{30} + 0.025550x^{36} + 0.095652x^{42} - 0.023675x^{48} \\ &- 0.011912x^{54} - 0.041730x^{60} + 0.089665x^{66} - 0.083628x^{72} \\ &+ 0.049185x^{78} - 0.019539x^{84} + 0.004901x^{90} - 0.000587x^{96}, \end{aligned} \tag{9}$$

with $\alpha = -1.4677424503$ [4,13].

The eigenproblem (2) has five relevant eigenvalues $\delta = 9.296246833, \alpha^2, \alpha^3, \alpha^4, \alpha^5$, so $\mathrm{CoDim}_\mathrm{S} = 5$. It agrees with a simple argument that to have an extremum of the sixth power in one-dimensional map we should satisfy four additional conditions (the second, third, fourth, and fifth derivatives must be zero), and the fifth parameter is necessary to ensure the situation of criticality.

However, in one-dimensional maps the S-type criticality may appear as a phenomenon of codimension 3 [4]. Suppose the map has a quadratic extremum and a cubic inflection point, and one of these points is mapped precisely to another. Then, the second iteration of the map has an extremum of the sixth power. In three-dimensional parameter space a curve may exist at which such a condition is valid. If the period-doubling cascade occurs along this curve, the accumulation point will be the critical point of S-type.

In the model map (6) we can satisfy the condition of mapping the extremum to the inflection point and find the critical point S_1: $A_\mathrm{c} = 1.8724481923$, $B_\mathrm{c} = -1.6252052847$, $C_\mathrm{c} = 1.0940161015$. Alternatively, if the inflection point is mapped to the extremum, we find the critical point S_2: $A_\mathrm{c} = 1.3799094808$, $B_\mathrm{c} = -0.5574097012$, $C_\mathrm{c} = 1.1818211223$. In both cases, there are three relevant eigenvalues. Being enumerated in order of decreasing absolute value, they are $\delta_1 = \delta$, $\delta_2 = \alpha^4$, $\delta_3 = \alpha^2$ for S_1, and $\delta_1 = \delta$, $\delta_2 = \alpha^3$, $\delta_3 = \alpha^2$ for S_2.

At the critical point S we have unstable cycles of all periods 2^k, and the universal multiplier is $\mu = -2.32150547$. Hausdorff dimension of the critical attractor is $D_0 = 0.68330$.

To write out relations for the parameter space scaling coordinates in a vicinity of a critical point S, we must account some nonlinear terms. The explicit expressions for parameters of the map (6) via scaling coordinates (C_1, C_2, C_3) are the following [4]:
for S_1:

$$\begin{aligned} A - A_\mathrm{c} &= -0.52856711(C_1 + 0.780253C_3^2) - 0.80973728C_2 + C_3, \\ B - B_\mathrm{c} &= (C_1 + 0.780253C_3^2) + C_2 - 0.848712657C_3, \\ C - C_\mathrm{c} &= -0.12666138(C_1 + 0.780253C_3^2) + 0.08221729C_2 + 0.59074778C_3, \end{aligned} \tag{10}$$

for S_2:

$$\begin{aligned} A - A_\mathrm{c} &= 0.83522562(C_1 - 0.0824C_3^2 + 0.1898C_2C_3) - 0.94065706C_2 + C_3, \\ B - B_\mathrm{c} &= -0.06885319(C_1 - 0.0824C_3^2 + 0.1898C_2C_3) + C_2 - 0.63818335C_3, \\ C - C_\mathrm{c} &= (C_1 - 0.0824C_3^2 + 0.1898C_2C_3) - 0.14833524C_2 - 0.07558422C_3, \end{aligned} \tag{11}$$

Figs. 2 and 3 show cross-sections of the parameter space by coordinate surfaces (C_i, C_j) and illustrate their scaling properties near the critical points S_1 and S_2, respectively.

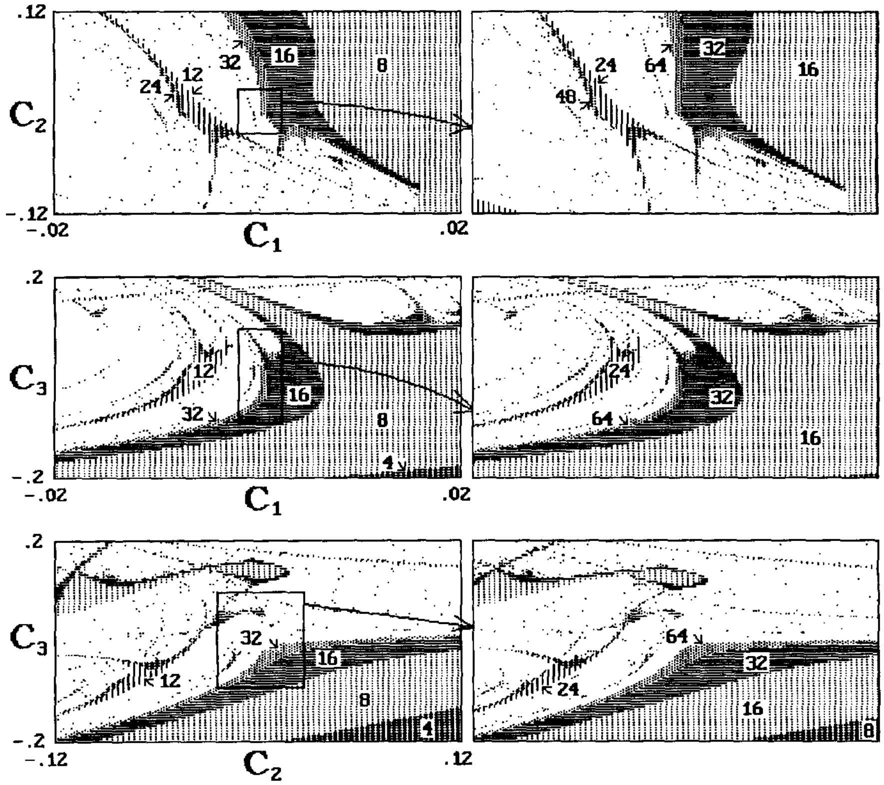

Fig. 2. Topography of the parameter space cross-sections near the critical point S_1 for the map (6). The scaling coordinates (10) are used. The critical point is located in the center of each picture.

2.4. *E-type criticality ("eight-power")*

The E-type criticality is associated with a solution of the RG equation represented by expansion in powers of x^8:

$$\begin{aligned} g(x) = 1 &- 1.897353x^8 - 0.738844x^{16} + 0.989783x^{24} + 0.445691x^{32} \\ &- 0.585998x^{40} - 0.281968x^{48} + 0.394947x^{56} - 0.032153x^{64} \\ &+ 0.391490x^{72} - 1.216074x^{80} + 1.458595x^{88} - 0.973413x^{96} \\ &+ 0.387179x^{104} - 0.086728x^{112} + 0.008478x^{120} \end{aligned} \qquad (12)$$

with $\alpha = -1.358017279$ [4,13].

The eigenproblem (2) has seven relevant eigenvalues $\delta = 10.94862427, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7$, so $\mathrm{CoDim_E} = 7$. This number may be explained in the same manner as it was done for the types T and S. However, we can argue that actually the E-type critical behavior may occur in one-dimensional maps as a codimension-3 phenomenon [4]. If one-dimensional map has three quadratic extrema, x_1 mapped to x_2, and x_2 mapped to x_3, the iterated map will have an extremum of the eighth power. In three-dimensional parameter space these conditions may be valid along some curve. If the period-doubling cascade occurs at this curve, the limit will be the critical point of E-type. Only three of seven eigenvalues are relevant in this case: $\delta_1 = \delta$, $\delta_2 = \alpha^4$, $\delta_3 = \alpha^2$. For the model map (6) such a critical point is located at $A_c = 2.4493669341$, $B_c = -1.2604157306$, $C_c = 0.7009546250$.

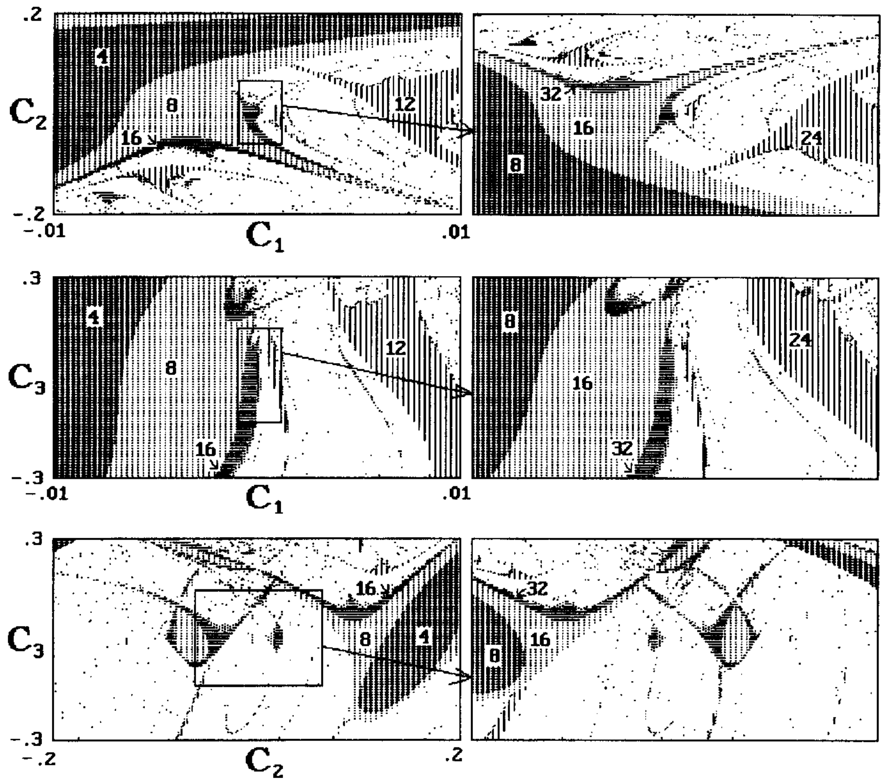

Fig. 3. Topography of the parameter space cross-sections near the critical point S_2 for the map (6). The scaling coordinates (11) are used. The critical point is located in the center of each picture.

The universal multiplier for unstable cycles of large periods 2^k is $\mu = -2.51408835$, and Hausdorff dimension of the critical attractor is $D_0 = 0.70750$.

The explicit expressions for parameters of the map (6) via scaling coordinates (C_1, C_2, C_3) near the E-type critical point are the following [4]:

$$\begin{aligned}
A - A_c &= (C_1 - 0.200585C_3^2 + 0.3443C_2C_3 - 0.13839C_3^3) + C_2 - 0.945646C_3, \\
B - B_c &= -0.559738(C_1 - 0.200585C_3^2 + 0.3443C_2C_3 - 0.13839C_3^3) - 0.4571C_2 + C_3, \\
C - C_c &= -0.338842(C_1 - 0.200585C_3^2 + 0.3443C_2C_3 - 0.13839C_3^3) + 0.31758C_2 - 0.00652C_3.
\end{aligned} \tag{13}$$

Fig. 4 shows cross-sections of the parameter space by the coordinate surfaces (C_i, C_j), and illustrates their scaling properties.

In conclusion we note that among the discussed critical situations only the classic Feigenbaum type has always the same codimension (1) both for one-dimensional maps and in general case. If one turns to multi-parameter analysis, the question becomes more subtle. For one-dimensional maps critical behavior of type T may be found in two-parameter families, S- and E-types – in three-parameter families, although their codimensions calculated from the RG analysis are higher. Formal reason is that one-dimensional maps represent a particular, non-typical class of dynamical systems. In maps of higher dimension, or in differential equations, the types T, S, and E will appear as phenomena of codimension 3, 5, and 7, respectively. In lower codimensions they may serve only for approximate description of realistic systems (as long as one-dimensional map remains a satisfactory model).

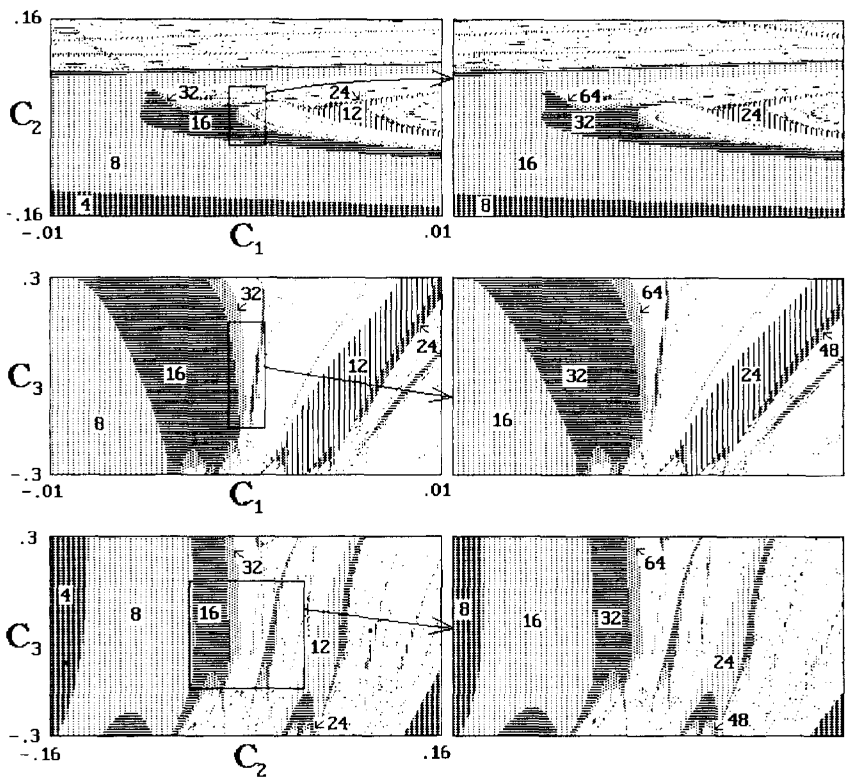

Fig. 4. Topography of the parameter space cross-sections near the E-type critical point for the map (6). The scaling coordinates (13) are used. The critical point is located in the center of each picture.

However, it was discovered recently, that in two-parameter analysis of a realistic model the tricritical scaling properties may be valid up to a surprisingly high order of period-doubling ("pseudo-tricritical behavior") [16]. In a sense, tricriticality appears as a kind of intermediate asymptotics, and it may be very difficult, if possible at all, to detect any deflection from true tricriticality in experiment or in straightforward computer calculations. We believe that in the same sense, as intermediate asymptotics, the scaling behavior of S- and E-types may be found in three-parameter analysis of realistic nonlinear dissipative systems.

3. Types of critical behavior for systems with uni-directional coupling

Now we start to consider the types of critical behavior which arise due to presence of an additional dimension; they do not occur in one-dimensional maps.

In Section 3 we turn to a special class of systems that can be decomposed into two elements with uni-directional coupling. Appropriate model maps may be taken in the form $x_{n+1} = F_1(x_n)$, $y_{n+1} = F_2(x_n, y_n)$. Formally speaking, the situation of uni-directional coupling is rather atypical, and the associated types of critical behavior will have high codimensions in the entire class of general two-dimensional maps. However, the uni-directional coupling appears to be physically realizable [16,18,19], and it justifies a special attention to this kind of systems.

In the case of the uni-directional coupling, the RG analysis leads to the following set of equations for the pair of functions g and f [20,21]:

$$g(x) = \alpha g(g(x/\alpha)), \tag{14}$$

$$f(x, y) = \beta f(g(x/\alpha),\ f(x/\alpha, y/\beta)). \tag{15}$$

The first relation coincides with the equation of Feigenbaum–Cvitanović. Thus, the problem consists in analysis of the second equation. Both the function $f(x, y)$ and the orbit scaling factor β must be found from this equation, while the function $g(x)$ is known.

In the case of uni-directional coupling, each type of critical behavior is characterized by two universal multipliers associated with two subsystems. These are the derivatives of functions $g(x)$ and $f(x, y)$ calculated at the fixed point $x_\star = g(x_\star)$, $y_\star = f(x_\star, y_\star)$: $\mu_1 = g'(x_\star)$, $\mu_2 = f'_y(x_\star, y_\star)$.

For a solution of the RG equations (14) and (15) we may consider a class of perturbations that do not violate the uni-directional nature of coupling. The associated eigenproblem can be solved separately for two subspaces: (i) in presence, and (ii) in absence of a parameter perturbation in the first subsystem. The eigenvalues of the first class are obtained from Eq. (2). For the second class, the following equation is valid [20]:

$$\nu v(x, y) = \beta[f'(g(x/\alpha), f(x/\alpha, y/\beta))v(x/\alpha, y/\beta) + v(g(x/\alpha), f(x/\alpha, y/\beta))], \tag{16}$$

where f' means the derivative with respect to the second argument.

The number of relevant eigenvalues both from Eqs. (2) and (16) defines codimension of the criticality type with respect to the class of systems with uni-directional coupling.

If we account perturbations associated with backward coupling, the linearized RG equations give rise to one more class of eigenfunctions:

$$\nu u(x, y) = \alpha[g'(g(x/\alpha))u(x/\alpha, y/\beta) + u(g(x/\alpha), f(x/\alpha, y/\beta))]. \tag{17}$$

Adding the relevant eigenvalues of the latter problem to those from from Eqs. (2) and (16), one obtains a total codimension of the criticality type in the entire class of general two-dimensional maps.

3.1. Double Feigenbaum point (type DF)

This type is associated with the solution of Eqs. (14) and (15) constructed of a pair of Feigenbaum's functions (3):

$$\{g, f\} = \{g_F(x), g_F(y)\} \tag{18}$$

with scaling constants $\alpha = \beta = \alpha_F = -2.5029\ldots$

Eq. (2) gives rise to one relevant eigenvalue $\delta = 4.6692\ldots$ Eigenfunctions of Eq. (16) may depend or not depend on x. Among the first class there is one relevant solution with eigenvalue $\delta = 4.6692\ldots$ The second class contains two solutions with $|\nu| > 1$: $\nu_1 = \alpha = -2.5029\ldots$ and $\nu_2 = 2$ [22]. Thus, the codimension in the class of systems with uni-directional coupling is 4. Accounting the eigenvalues for backward coupling one finds total codimension: $\mathrm{CoDim}_{\mathrm{DF}} = 6$. The set of relevant eigenvalues includes $\delta_{1,2} = \delta_F$, $\delta_{3,4} = \alpha_F$, $\delta_{5,6} = 2$.

Let us consider a model system of two logistic maps with uni-directional coupling [20,21]:

$$x_{n+1} = 1 - \lambda x_n^2, \qquad y_{n+1} = 1 - Ay_n^2 - Bx_n^2, \tag{19}$$

where x and y are dynamical variables for the first and second subsystems, λ, A, and B are the control parameters. The critical point DF is located at $\lambda_c = A_c = 1.401155\ldots$, $B_c = 0$.

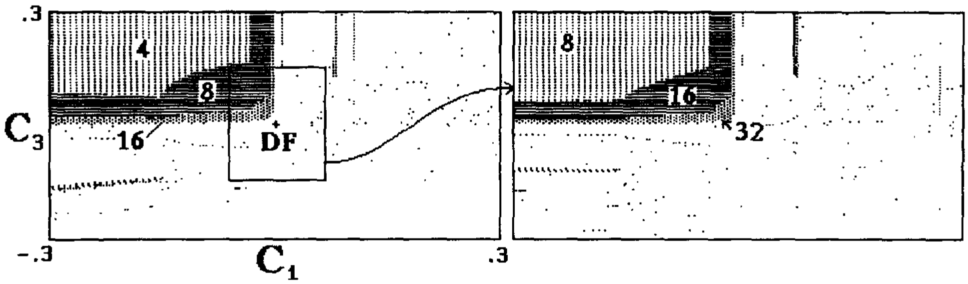

Fig. 5. The parameter space topography for the model map (19) with unidirectional coupling near the double Feigenbaum point in the coordinate plane (C_1, C_3) (see (20)).

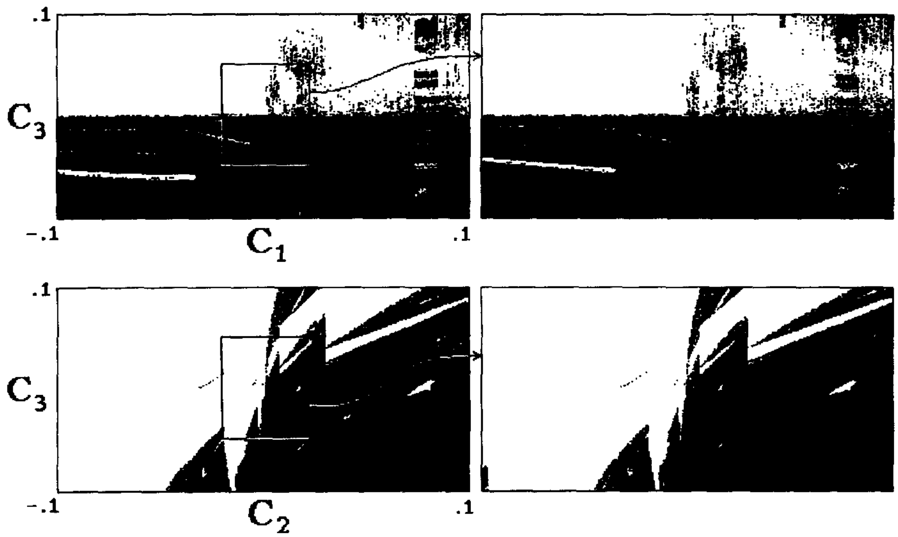

Fig. 6. Domains of positive Lyapunov exponent in the second subsystem (black) for the model map with uni-directional coupling (19). The pictures relate to the neighborhood of the double Feigenbaum point and show cross-sections of the parameter space by the coordinate surfaces defined according to Eqs. (20).

At the point DF we have two uncoupled one-dimensional maps, so both x and y run over a Feigenbaum attractor. The whole system has an infinite number of attractors because an arbitrary relative phase shift is possible between two subsystems.

The eigenvalues responsible for scaling properties of the three-dimensional parameter space of the model map (19) are $\delta_1 = \delta_F$, $\delta_2 = \delta_F$, $\delta_3 = 2$. (The eigenfunction with $\nu = -2.5029\ldots$ contains non-zero odd part, so it does not arise when we perturb the parameters in (19).)

The appropriate scaling coordinate system (C_1, C_2, C_3) may be defined by the relations

$$\lambda - \lambda_c = C_1, \quad A - \lambda_c = C_2 - C_3, \quad B = C_3. \tag{20}$$

In Fig. 5 we show domains of periodicity in the coordinate plane (C_1, C_3) and demonstrate its scaling property. For other cross-sections of the parameter space there are no notable regions of periodicity. In Fig. 6 domains of chaos (positive Lyapunov exponent) in the second subsystem are shown as black in the cross-sections of the parameter space by the coordinate surfaces.

3.2. Bicritical behavior (type B)

This type of behavior is realized in a system with uni-directional coupling when the first subsystem exhibits Feigenbaum criticality while the critical behavior of the second subsystem is associated with the following solution of Eq. (15):

$$\begin{aligned} f(x,y) = 1 &- 0.596905x^2 - 0.032157x^4 + 0.018457x^6 - 0.000201x^8 - 0.855639y^2 \\ &- 0.302943x^2y^2 + 0.054630x^4y^2 + 0.021499x^6y^2 - 0.004860x^8y^2 \\ &- 0.431738y^4 + 0.087452x^2y^4 + 0.091136x^4y^4 - 0.011023x^6y^4 \\ &- 0.003242x^8y^4 + 0.087486y^6 + 0.180356x^2y^6 + 0.009298x^4y^6 \\ &- 0.031914x^6y^6 + 0.005042x^8y^6 + 0.152662y^8 + 0.060337x^2y^8 \\ &- 0.09631x^4y^8 + 0.017439x^6y^8 + 0.060864y^{10} - 0.153737x^2y^{10} \\ &+ 0.03769x^4y^{10} - 0.101867y^{12} + 0.047570x^2y^{12} + 0.026310y^{14} \end{aligned} \tag{21}$$

with $\alpha = -2.502907876$ and $\beta = -1.505318159$ [20].

Numerical solution of Eqs. (2) and (16) yields two relevant eigenvectors. The vector with non-zero u-component corresponds to Feigenbaum eigenvalue $\delta_1 = 4.6692\ldots$ The second vector has $u \equiv 0$ and non-zero v-component; the eigenvalue is $\delta_2 = 2.39272443$. Thus, in the class of systems with uni-directional coupling the codimension of the bicritical situation is 2. If one wishes to account backward coupling, the eigenvalues given by Eq. (17) must be added: $\delta_3 = 4.296897$, $\delta_4 = \alpha^2/\beta = -4.161611$, $\delta_5 = \alpha^2/\beta^3 = -1.83648$, $\delta_{6,7} = 0.9404 \pm 0.4024i$. So, the total codimension is $\mathrm{CoDim_B} = 7$.

For the model map (19) a bicritical curve exists in the parameter space (λ, A, B). For particular value of coupling parameter $B = 0.375$ we have found the bicritical point at $\lambda_c = 1.401155189$, $A_c = 1.124981403$.

At the bicritical point unstable cycles of all periods 2^k exist. The universal asymptotic values of two multipliers are $\mu_1 = -1.6011913$ and $\mu_2 = -1.17885538$. Bicritical attractor is a fractal set embedded into (x, y)-plane; its Hausdorff dimension was evaluated as $D_0 = 1.0785$ [20].

Fig. 7 shows topography of the parameter plane (λ, A) near the bicritical point B of the model map (19). Scaling properties of the picture are illustrated using the natural parameters of the map (obviously, shifts of λ or A give rise to pure perturbations associated with δ_1 and δ_2, respectively).

3.3. Criticality of BT-type

This relates to bicriticality like tricriticality relates to Feigenbaum behavior. Again, the first subsystem is supposed to be in the Feigenbaum critical state, but the solution of Eq. (15) has expansion in powers of y^4, and the scaling factors are $\alpha = \alpha_F = -2.5029\ldots$ and $\beta = -1.2416604$ [21].

Relevant eigenvalues for Eqs. (2) and (16) are now $\delta_1 = 4.669201$, $\delta_2 = 2.654654$, $\delta_3 = \beta^3 = -1.9142934$, $\delta_4 = \beta^2 = 1.5417206$. Thus, the codimension of the BT-criticality in the class of systems with uni-directional coupling is 4. Total codimension of the BT point remains unknown due to technical difficulties of accurate estimation of all relevant eigenvalues responsible for backward coupling. Certainly, it is very high.

In the model map (19) the BT critical point may be found by appropriate choice of three parameters: $\lambda_c = 1.401155$, $A_c = 1.066053$, $B_c = 0.835050$. In this case, the fourth-power order of y appears after two iterations due to mapping an extremum to an extremum.

In the model map

$$x_{n+1} = 1 - \lambda x_n^2, \qquad y_{n+1} = 1 - P y_n^4 - Q x_n^2, \tag{22}$$

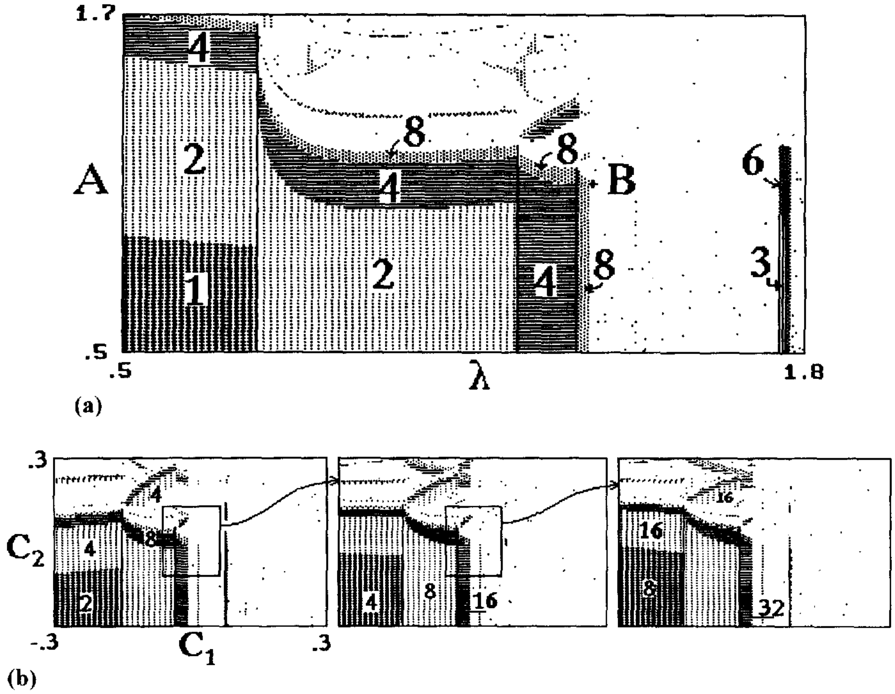

Fig. 7. Topography of the parameter plane (λ, A) for the model map (19), $B = 0.375$ (a), and illustration of scaling properties near the bicritical point (b). Scaling coordinates are simply $C_1 = \lambda - \lambda_c$ and $C_2 = A - A_c$.

we have a critical curve BT in the parameter space (λ, P, Q). For a fixed Q, the point BT may be found by tuning two parameters, λ and P. In particular, for $Q = 0.375$ the critical point is located at $\lambda_c = 1.401155$, $P_c = 1.279735$. The eigenvalues responsible for the scaling properties of the parameter plane (λ, P) are δ_1 and δ_2.

At the BT point there exist unstable cycles of all periods 2^k, and the universal values of two multipliers are $\mu_1 = -1.601191$ and $\mu_2 = -1.398015$. Attractor is a fractal object analogous to the bicritical attractor but with distinct quantitative characteristics.

The topography of the parameter plane (λ, P) for the map (22) is shown in Fig. 8, and the scaling properties intrinsic to a vicinity of the critical point BT are illustrated.

4. Types of critical behavior intrinsic to two-dimensional maps

To consider types of critical behavior that arise in general case due to involving an additional phase space dimension, we need the two-dimensional version of the Feigenbaum–Cvitanović RG equation. The generalization is rather straightforward, if we suppose that a special coordinate system exists in the two-dimensional phase space, the requirement is that the scale change during the RG transformation must be "diagonal": ($X \to X/\alpha$, $Y \to Y/\beta$). If the critical behavior is associated with a fixed point of the RG transformation, the equations look like

$$\begin{aligned} g(X, Y) &= \alpha g(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta)), \\ f(X, Y) &= \beta f(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta)) \end{aligned} \tag{23}$$

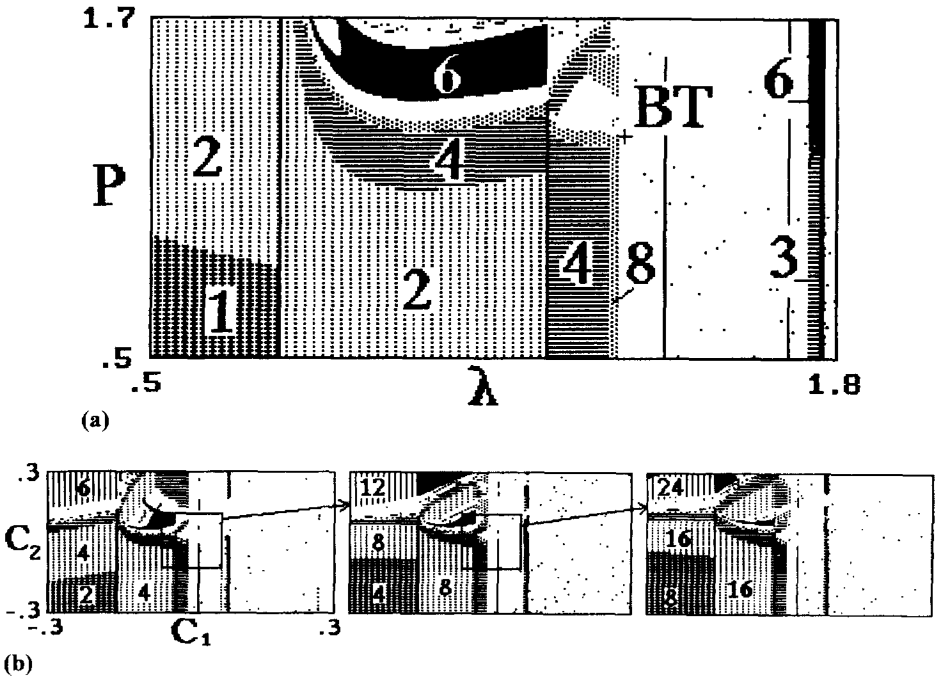

Fig. 8. Topography of the parameter plane (λ, P) for the model map (22), $Q = 0.375$ (a), and illustration of scaling properties near the critical point BT (b). Scaling coordinates are $C_1 = \lambda - \lambda_c$ and $C_2 = P - P_c$.

with two scaling factors α and β to be found (see [26,28]). We emphasize that the "scaling variables" X, Y usually will not coincide with the "natural" variables of model maps.

The two-dimensional generalization of the eigenproblem for the linearized RG equation at the fixed point (g, f) is

$$\begin{aligned}\nu u(X,Y) &= \alpha[g_1'(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))u(X/\alpha, Y/\beta) \\ &\quad + g_2'(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))v(X/\alpha, Y/\beta) \\ &\quad + u(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))], \\ \nu v(X,Y) &= \beta[f_1'(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))u(X/\alpha, Y/\beta) \\ &\quad + f_2'(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))v(X/\alpha, Y/\beta) \\ &\quad + v(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta))],\end{aligned} \tag{24}$$

where subscripts 1 and 2 designate derivatives of the functions with respect to the first and second argument. Again, a number of relevant eigenvalues with $|\nu| > 1$ define codimension of the criticality type, and the eigenvalues themselves are scaling factors for appropriate directions in the parameter space.

The universal multipliers for the two-dimensional case are obtained as eigenvalues of the Jacobi matrix calculated at the fixed point $X_\star = g(X_\star, Y_\star)$, $Y_\star = f(X_\star, Y_\star)$.

4.1. Hamiltonian criticality (H-type)

This is a commonly known type of behavior discovered in area-preserving two-dimensional maps at the onset of chaos via period-doubling cascade [29,30]. It is associated with the following fixed point solution of the RG equation:

$$\begin{aligned} g(X,Y) = {} & 1 - 0.125219Y + 0.000394Y^2 - 0.194676X - 0.005005XY \\ & - 0.000011XY^2 - 0.914763X^2 + 0.005888X^2Y + 0.000008X^2Y^2 \\ & - 0.036937X^3 - 0.000160X^3Y + 0.021995X^4 + 0.000058X^4Y \\ & - 0.000591X^5 + 0.000003X^5Y + 0.000140X^6 - 0.000002X^6Y \\ & + 0.000008X^7 - 0.000005X^8, \\ f(X,Y) = {} & 1 - 2.055648Y + 0.119783Y^2 - 0.000724Y^3 + 4.790122X \\ & + 0.320433XY + 0.008078XY^2 - 0.000010XY^3 - 14.863762X^2 \\ & + 1.765345X^2Y - 0.015898X^2Y^2 + 0.000014X^2Y^3 + 2.340180X^3 \\ & + 0.118352X^3Y - 0.000220X^3Y^2 - 0.000004X^3Y^3 + 6.505040X^4 \\ & - 0.116336X^4Y + 0.000171X^4Y^2 + 0.000003X^4Y^3 + 0.433469X^5 \\ & - 0.001676X^5Y - 0.000039X^5Y^2 - 0.283749X^6 + 0.000923X^6Y \\ & + 0.000020X^6Y^2 - 0.004248X^7 - 0.000194X^7Y + 0.001855X^8 \\ & + 0.000075X^8Y - 0.000358X^9 + 0.000002X^9Y + 0.000111X^{10} \\ & - 0.000002X^{10}Y + 0.000003X^{11} - 0.000002X^{12}, \end{aligned} \tag{25}$$

with $\alpha = -4.0180767046$ and $\beta = 16.3638968792$ (see [23–27] for details, different versions, and hints for the RG calculations). Note that the Jacobi determinant for (g, f) is identically equal to unity.

The eigenproblem (24) gives two relevant eigenvalues: $\delta_1 = 8.7210972$ and $\delta_2 = 2$ [3,31]. The eigenvalue δ_1 is associated with a perturbation retaining the map inside the area-preserving class, and δ_2 is responsible for dissipation. Thus, for area-preserving maps the criticality of H-type is a phenomenon of codimension 1. However, it may arise also in dissipative systems, and in the last case it appears as a phenomenon of codimension 2, $\text{CoDim}_\text{H} = 2$.

As an example, let us consider Henon map

$$x_{n+1} = 1 - \lambda x_n^2 - by_n, \qquad y_{n+1} = x_n. \tag{26}$$

It is known that for $b < 1$ increasing λ one observes classic Feigenbaum period-doubling cascade (type F in our notation). When $b = 1$ the map becomes area-preserving. In the parameter plane (b, λ) at $b_c = 1$, $\lambda_c = 4.136166803904 28$ the critical point H is located. In other words, the border of chaos for Henon map is Feigenbaum critical curve terminated at $b = 1$ by the H-point (Fig. 9). (See [3,31] for discussion of crossover between these two types of criticality.)

Precisely at the critical point H there exists a set of unstable cycles of periods 2^k. The universal multipliers are $\mu_1 = -2.057478352$ and $\mu_2 = 1/\mu_1 = -0.486031845$. Due to conservative nature of the dynamics at the critical point, there is no attractor at this point. However, the state space has a self-similar arrangement. For the model map (26) one can observe this self-similarity in scaling variables $X = x - 0.0475282$, $Y = (1 - \lambda x^2)/2 - y$: the structure reproduces itself under magnification by factors α and β along the axes X and Y, respectively.

The scaling coordinates (C_1, C_2) for the model map (26) may be defined via the following relations:

$$\lambda - \lambda_c = C_1 + \lambda_c C_2 + 1.5600931C_2^2, \quad b = 1 + C_2. \tag{27}$$

In Fig. 9 the topography of dynamical regimes is shown in a neighborhood of the critical point H, and the scaling properties are illustrated.

4.2. Critical behavior of FQ-type ("Feigenbaum + quasiperiodicity")

The critical behavior of FQ-type is associated with the fixed point solution of the RG equation (23) containing powers of X^2 and XY in its expansion:

$$\begin{aligned} g(X,Y) = {} & 1 - 1.09789X^2 + 0.15708X^4 - 0.01760X^6 - 0.00656X^8 \\ & + 0.00223X^{10} - 0.71142XY + 0.08649X^3Y + 0.04397X^5Y \\ & - 0.01794X^7Y + 0.00349X^9Y + 0.00177X^2Y^2 + 0.03788X^4Y^2 \\ & - 0.01465X^6Y^2 + 0.00381X^8Y^2 - 0.00062X^{10}Y^2 + 0.00715X^3Y^3 \\ & - 0.00373X^5Y^3 + 0.00066X^7Y^3 - 0.00007X^9Y^3 - 0.00034X^4Y^4 \\ & - 0.00003X^6Y^4 + 0.00005X^8Y^4 - 0.00001X^{10}Y^4 - 0.00002X^5Y^5 \\ & + 0.00002X^7Y^5 - 0.00001X^9Y^5, \\ f(X,Y) = {} & 1 + 0.06696X^2 + 1.54157X^4 - 1.06106X^6 + 0.19682X^8 \\ & - 0.01239X^{10} - 2.79602XY + 1.36187X^3Y - 0.81578X^5Y \\ & + 0.17873X^7Y - 0.00300X^9Y + 0.21012X^2Y^2 - 0.15760X^4Y^2 \\ & - 0.01068X^6Y^2 + 0.06894X^8Y^2 - 0.02013X^{10}Y^2 - 0.00452X^3Y^3 \\ & - 0.01900X^5Y^3 + 0.03524X^7Y^3 - 0.01132X^9Y^3 - 0.00502X^4Y^4 \\ & + 0.01271X^6Y^4 - 0.00757X^8Y^4 + 0.00132X^{10}Y^4 + 0.00170X^5Y^5 \\ & - 0.00178X^7Y^7 + 0.00046X^9Y^5 - 0.00018X^6Y^6 + 0.00006X^8Y^6 \\ & + 0.00001X^{10}Y^6 + 0.00003X^7Y^7 - 0.00001X^9Y^7, \end{aligned} \tag{28}$$

the scaling factors are $\alpha = -1.90007167$ and $\beta = -4.00815785$ [5,28].

The eigenproblem (24) has three relevant eigenvalues $\delta_1 = 6.32631925$, $\delta_2 = 3.44470967$, and $\delta_3 = \alpha = -1.90007167$. Hence, $\mathrm{CoDim}_{\mathrm{FQ}} = 3$.

An appropriate model map is

$$x_{n+1} = 1 - ax_n^2 + dx_ny_n, \quad y_{n+1} = 1 - bx_ny_n. \tag{29}$$

In the three-dimensional parameter space (a, b, d) a curve exists which is a locus of the FQ points. For $d = 0.3$ the critical point FQ is located at $a_c = 1.767192895$, $b_c = 1.629678013$. It is obtained as a limit for a sequence of the terminal points for period-doubling bifurcation lines, where both multipliers of a cycle become equal to -1 [5].

At the critical point FQ we have an infinite set of unstable cycles of period 2^k. The asymptotic values of the multipliers are $\mu_1 = -1.579739$ and $\mu_2 = -1.057149$. Critical attractor is a fractal, and it may be imagined as a limit object – "period-2^∞ orbit". Self-similarity of the attractor may be observed in scaling coordinate system $X = x$, $Y = y - 2.1091x$, under scale change with the factors α and β along the axes X and Y, respectively.

Due to a specific form of the model map (29), the point FQ was found in two-parameter analysis. In this map only the eigenvalues δ_1 and δ_2 are presented in perturbations arising due to a shift of the parameters from the critical point. Topography of the parameter plane and its scaling properties are illustrated in Fig. 10. The scaling coordinates are defined via the expressions

$$a - a_c = C_1 + 0.47733C_2, \quad b - b_c = C_2. \tag{30}$$

A model map with a number of parameters equal to the total codimension may be written in the following form [5]:

$$x_{n+1} = 1 - ax_n^2 + d(x_n - c)y_n, \quad y_{n+1} = 1 - b(x_n - c)y_n. \tag{31}$$

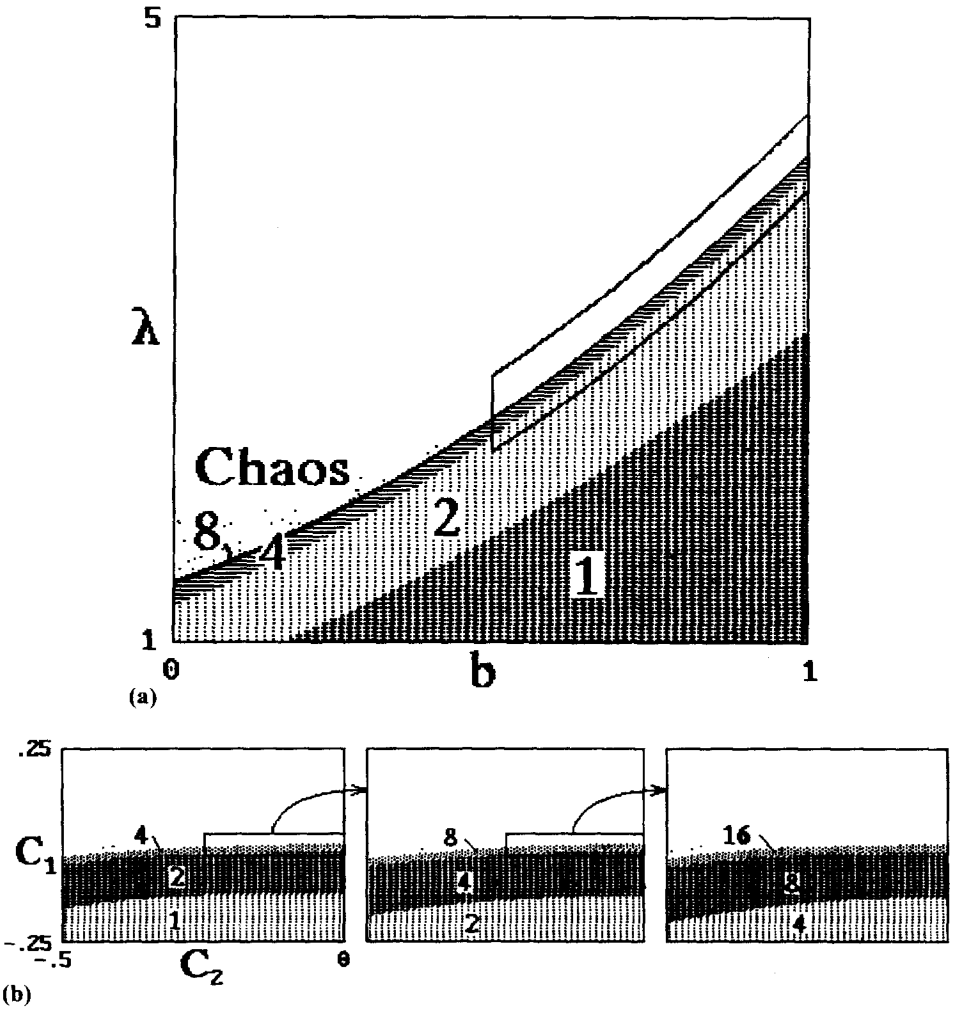

Fig. 9. Topography of the parameter plane (b, λ) for Henon map (26) (a), and illustration of scaling properties near the Hamiltonian critical point H (b). Scaling coordinates are defined by (27). The curvilinear quadrangle in (a) corresponds to the region seen in the first picture (b).

It allows to study universal arrangement of three-dimensional vicinity of the FQ-point in the parameter space (a, d, c).

4.3. Critical behavior of C-type ("cycle")

The critical behavior of C-type is associated with a cycle of period 2 for the RG transformation (23) [5,28]. The solution consists of two pairs of functions:

$$\begin{aligned}
g_2(X, Y) &= \alpha_1 g_1(g_1(X/\alpha_1, Y/\beta_1), f_1(X/\alpha_1, Y/\beta_1)),\\
f_2(X, Y) &= \beta_1 f_1(g_1(X/\alpha_1, Y/\beta_1), f_1(X/\alpha_1, Y/\beta_1)),\\
g_1(X, Y) &= \alpha_2 g_2(g_2(X/\alpha_2, Y/\beta_2), f_2(X/\alpha_2, Y/\beta_2)),\\
f_1(X, Y) &= \beta_2 f_2(g_2(X/\alpha_2, Y/\beta_2), f_2(X/\alpha_2, Y/\beta_2)).
\end{aligned} \tag{32}$$

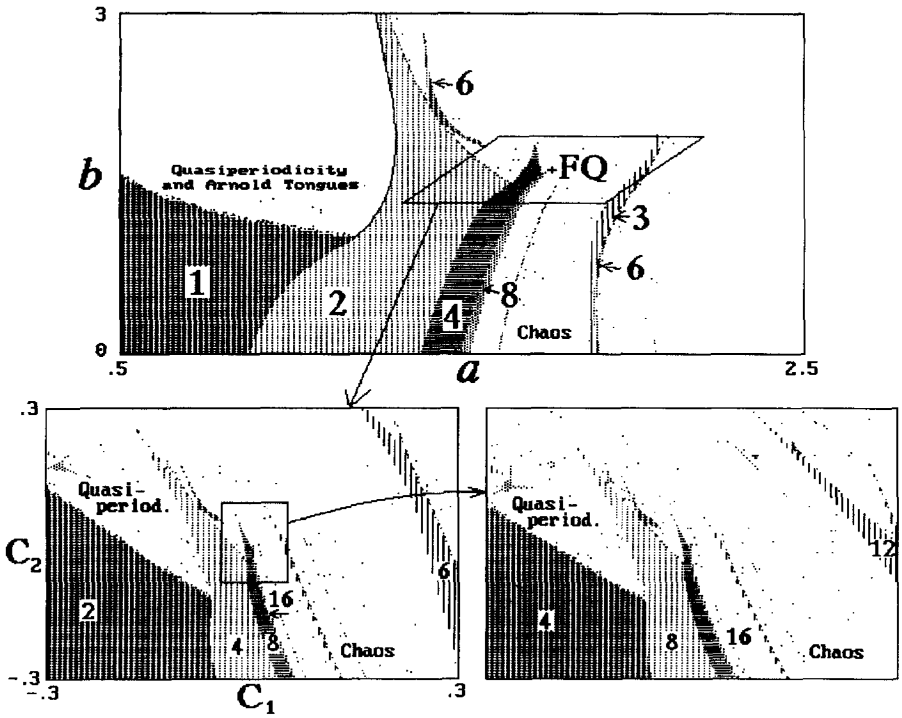

Fig. 10. Topography of the parameter plane (a, b) for the map (29), $d = 0.3$. The neighborhood of the critical point FQ is shown separately using the scaling coordinates (30).

We present here the polynomial expansions only for one pair:

$$\begin{aligned}
g_1(X, Y) = {} & 1 - 1.27700X^2 + 0.08729X^4 + 0.00628X^6 - 0.00077X^8 + 0.00002X^{10} \\
& - 0.49955Y + 0.13906X^2Y + 0.00422X^4Y - 0.00178X^6Y \\
& + 0.00012X^8Y + 0.04634Y^2 - 0.00148X^2Y^2 - 0.00142X^4Y^2 \\
& + 0.00018X^6Y^2 - 0.00108Y^3 - 0.00045X^2Y^3 + 0.00012X^4Y^3 \\
& - 0.00001X^6Y^3 - 0.00004Y^4 + 0.00004X^2Y^4 - 0.00001X^4Y^4, \\
f_1(X, Y) = {} & 1 - 2.32102X^2 + 0.39699X^4 + 0.0017X^6 - 0.00489X^8 \\
& + 0.00043X^{10} + 0.22671Y + 0.50507X^2Y - 0.03003X^4Y \\
& - 0.00829X^6Y + 0.00144X^8Y - 0.00003X^{10}Y - 0.00001X^{12}Y \\
& + 0.14401Y^2 - 0.03241X^2Y^2 - 0.00414X^4Y^2 + 0.00173X^6Y^2 \\
& - 0.00012X^8Y^2 - 0.00001X^{10}Y^2 - 0.00869Y^3 - 0.00035X^2Y^3 \\
& + 0.00096X^4Y^3 - 0.00014X^6Y^3 + 0.00014Y^4 + 0.00024X^2Y^4 \\
& - 0.00008X^4Y^4 + 0.00002Y^5 - 0.00002X^2Y^5.
\end{aligned} \tag{33}$$

Note that relations (1) and (24) were called the "doubling transformations" [1,26]. We adapt the terminology and say that the function pairs g_1, f_1 and g_2, f_2 are the fixed points of the "quadrupling RG transformation". The scaling factors defined for this quadrupling transformation are $\alpha = \alpha_1\alpha_2 = 6.565350$ and $\beta = \beta_1\beta_2 = 22.120227$.

Using the numerical solution (33) we have found that the map $(x, y) \to (g_1(x, y), f_1(x, y))$ has a stable fixed point $X_\star = 0.25039$, $Y_\star = 1.59489$ with multipliers $\mu_1 = -0.725255$ and $\mu_2 = 0.847450$. Recall, however, that the map (g_1, f_1) being successively iterated four times, and then rescaled $(X \to X/\alpha, Y \to Y/\beta)$, turns to itself again. Therefore, presence of the stable fixed point implies existence of stable cycles of periods 4^k, $k = 1, 2, \dots, \infty$, all with the same multipliers. (Note that one point at an orbit of period-4^k may be easily found: $X_\star/\alpha^k$, $Y_\star/\beta^k$.)

We conclude that the map $(x, y) \to (g_1(x, y), f_1(x, y))$ defined by Eq. (33) exhibits an infinite (denumerable) set of coexisting attractors – the cycles of period 4^k. We call this set *the critical quasi-attractor* [32]. Usually a quasi-attractor is regarded as a complex object in phase space containing an infinite number of stable periodic orbits, whereas the empirically observable behavior seems to be chaotic [19]. In our case, we talk about the phenomenon that occurs at the border of chaos, admits an RG analysis, and exhibits the properties of quantitative universality and scaling.

For the map $(x, y) \to (g_2(x, y), f_2(x, y))$ one can find an unstable fixed point with multipliers $\mu_1 = -0.848865$ and $\mu_2 = 1.174459$, and a stable period-2 cycle. Thus, critical quasi-attractor consists of the stable cycles of periods 2×4^k.

By linearizing (32) one can derive the eigenproblem for perturbations of the solution under the quadrupling RG transformation. The largest eigenvalues (excluding those associated with infinitesimal variable changes) are $\delta_1 = 92.43126348$, $\delta_2 = 4.19244418$, $\delta_3 = 0.93$. Only δ_1 and δ_2 are larger than unity, so formally we conclude that $\mathrm{CoDim}_C = 2$. However, δ_3 is not far from unity, and, in general, this eigenvalue is responsible for a very slow convergence: usually the quantitative universality of the C-type will manifest itself only after a very large number of the period-doubling bifurcations.

For a model map [32]

$$x_{n+1} = a - x_n^2 + by_n, \quad y_{n+1} = -x_n^2 + dy_n, \tag{34}$$

we have selected numerically $b = -0.6663$ to ensure fast convergence (at this value of b the weakly decaying eigenmode in the solution of the RG equation is eliminated). First, we have found numerically the sequence of the codimension-2 bifurcation points – the terminal points of the period-doubling curves, where two multipliers become equal to -1 and 1. Then, the critical point C is found as the limit point of this sequence: $a_c = 0.24990280$, $d_c = 0.45290288$ [5]. At this point the critical quasi-attractor consists of stable cycles of periods 4^k. This set has a property of self-similarity, and the scaling variables are $X = x$, $Y = y + 1.31644753$.

Parameter space scaling coordinates (C_1, C_2) for the map (34) are related to the original parameters via the equations

$$a - a_c = C_1 - C_2 - 1.546069C_2^2 - 2.15C_2^3, \qquad d - d_c = 0.79016607C_2. \tag{35}$$

In Fig. 11 we show topography of the parameter plane for the map (34). In Fig. 12 the scaling coordinates (35) are used to show stability domains for orbits of period 4^k and 2×4^k. Observe self-similarity of the parameter plane arrangement under magnification by the factors δ_1 and δ_2 along the respective coordinate axes.

The C-type critical point may be found also for positive b. The best convergence occurs at $b = 0.6544$, and the critical point is located at $a_c = 0.566620683$, $d_c = 1.597132592$. At this point the cycle of the RG equation oscillates in the opposite phase, and the critical quasi-attractor consists of stable cycles of periods 2×4^k.

5. Conclusion

It was a remarkable general idea of Poincaré to study and classify mathematical entities according to their codimension. First, one considers the generic cases, then the situations typical in presence of one control parameter,

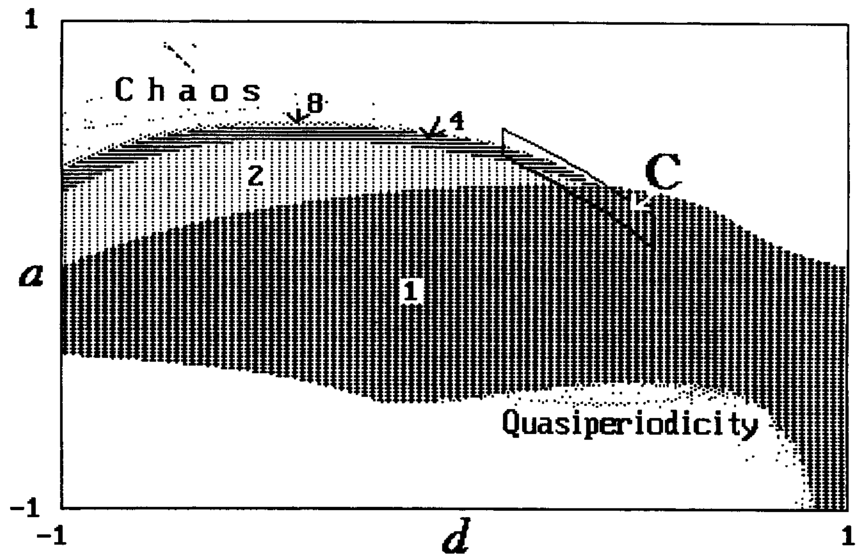

Fig. 11. Topography of the parameter plane (a, d) for the map (34), $b = -0.6663$. A curvilinear quadrangle corresponds to the region shown in the top picture of Fig. 12.

then the situations typical in two-parameter families, and so forth. In particular, this approach is exploited in bifurcation theory and catastrophe theory. We believe that the same idea may serve as a basis for a classification of the critical situations at the chaos boundary. We may call this field *the theory of multi-parameter criticality*. The maps listed in Sections 2–4 represent certain classes of quantitative universality and play for these classes the same role as the logistic map plays for Feigenbaum critical behavior. In appropriate cases, these maps may be useful for phenomenological description of multi-parameter systems near the onset of chaos via period-doubling.

In the present paper we restrict ourselves with the period-doubling situations associated with fixed points of the doubling and quadrupling RG transformations. Certainly, it is only a little part of some great entire picture. First, we are not sure that our list of the low-codimensional period-doubling critical situations is complete. Second, beside the period-doubling cascades there exist other types of transition to chaos, say, via intermittency and quasi-periodicity (see [33,34]). Third, it is known that critical behavior at the onset of chaos may be associated not only with fixed points, but also with more complicated periodic and non-periodic saddle orbits generated by the RG equations [33,35–41].

Nevertheless, the presented material obviously gives a perspective for investigation of non-Feigenbaum period-doubling critical situations in systems of different physical nature, in particular, in experiments.

Since now, only Feigenbaum's type of critical behavior was found and studied in details for many systems (see e.g. [42]).

As to other types of the period-doubling critical behavior, it is a separate interesting problem to find and investigate them in real systems, or in realistic models, such as differential equations. Certainly, this is a more difficult task than observation of classic Feigenbaum behavior, because the codimensions are higher, and many subtle details should be taken into account. However, we believe that in carefully organized natural and computer experiments some of the discussed types of criticality could be detected.

In two-dimensional diagrams of parameter planes for period-doubling systems of different nature (e.g. nonlinear oscillators with periodic excitation [44,45], modulated NMR laser [46], electronic self-oscillating systems [15,19], and so forth) one can observe often a structure called "crossroad area" [43]. This is just the region of the parameter

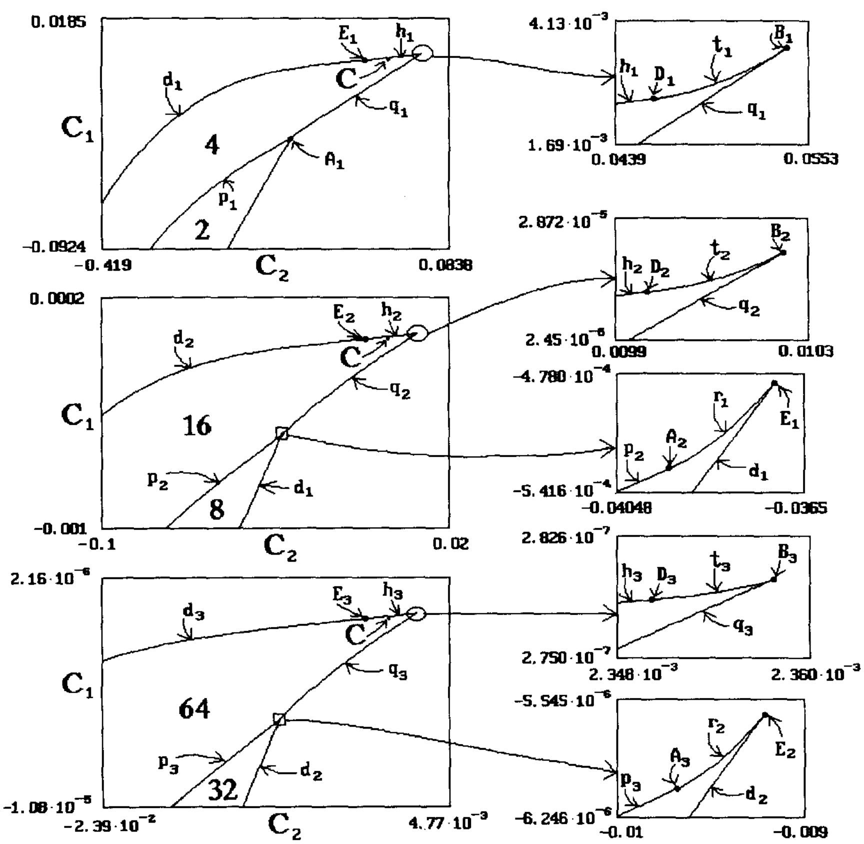

Fig. 12. Stability domains for cycles of periods 2, 4, ..., 64 for the map (34) near the C-type critical point in the scaling coordinates (35). For each subsequent picture the magnification is increased by $\delta_1 = 92.4312$ along the axis C_1, and by $\delta_2 = 4.1924$ along the axis C_2. Letters p and d denote the bifurcation curves for birth and stability loss of a cycle via period-doubling, q – the Neimark bifurcation, h – "hard" flip bifurcation, t and r – saddle–node bifurcations.

space where the (pseudo-)tricritical behavior should be expected. Indeed, for one concrete electronic system this behavior was found numerically in realistic model and studied in some details [16].

Many types of critical behavior, including bicriticality B, occur in dissipative period-doubling systems with uni-directional coupling (see, e.g. [16,18,47,48]). If one adds backward coupling, it becomes possible to realize criticality of types C, FQ, and H [28].

One more possibility for criticality of C-type arise if we consider a low-dimensional dissipative chaotic system synchronized by external periodic force. If the period-doubling bifurcation curves meet the edge of Arnold tongue [19], we conjecture that the C-type criticality may be expected there.

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