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CRITICAL DYNAMICS OF PITCH-FORK BIFURCATION IN A SYSTEM DRIVEN BY A FRACTAL SEQUENCE

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We consider the system, exhibiting pitch-fork bifurcation, forced by the external perturbation with fractal properties. Developing a renormalization group approach, we show that the situation is characterized by nonclassical critical exponents. These exponents appear to depend on external influence intensity and we get the analytical expressions for them. Apparently, a new kind of strange nonchaotic attractor arises in the case under consideration.

1. Introduction

One of the interesting general problems in nonlinear science is the dynamics of systems forced by external perturbations that may be periodic, quasiperiodic or chaotic. Depending on the system under consideration and the nature of the external force, one can observe many remarkable phenomena such as mode locking, quasiperiodicity, chaos, strange nonchaotic attractors, chaotic synchronization and so on.

In this note we consider the system with pitch-fork bifurcation driving by the signal with fractal properties. The criticality in this system is studied using the renormalization group (RG) approach. We obtain the analytical expressions for two critical exponents; one of them characterizes the dependence of the dynamical variable on the parameter of supercriticality near the critical point, and the other one describes the time decay of this variable at the critical point. These exponents appear to be different from classical values and depend on the intensity of external force.

2. Model System

The system under our consideration in the absence of an external force exhibits a pitch-fork bifurcation:

$$x_{n+1} = \Lambda \frac{x_n}{\sqrt{1+x_n^2}}\,, \tag{1}$$

where x_n is the dynamical variable value at the nth step of discrete time, and Λ is the control parameter.

The pitch-fork bifurcation occurs in system (1) at the critical parameter value $\Lambda_c = 1$. For stationary state we substitute $x_n = x_{n+1} = C$ and obtain $C \propto (\Lambda - \Lambda_c)^\beta$, where $\beta = 1/2$ is the classical critical exponent. One can find easily that temporal dependence of x_n at the critical point $\Lambda = \Lambda_c$ is characterized by the power law $x \propto n^\gamma$ that contains another classical critical exponent $\gamma = -1/2$.

Let us now assume that in Eq. (1) the control parameter Λ depends on n. A particular example of such problem was discussed by Kuznetsov *et al.* [1995]. The quasiperiodic external force was chosen: $\Lambda_c \propto \sin(2\pi(\omega n+\vartheta))$, where $\omega = (\sqrt{5}-1)/2$ (the golden mean). One could observe the birth of a strange nonchaotic attractor in this system and the main subject of the referred article was RG analysis of this criticality. Note that nonclassical critical behavior was observed in this case; in particular, dependence of the critical exponents on the phase variable ϑ was detected.

We shall consider another kind of driving force. This force with fractal properties is defined via Logistic Sign–Sequence (LSS) which is the sequence of signs of the dynamical variable of the logistic map at the period-doubling bifurcations accumulation point [Procaccia *et al.*, 1987].

$$\begin{aligned} \xi_n &= \mathrm{sign}(y_n) \\ y_{n+1} &= 1 - \lambda_c y_n^2 \\ y_0 = 0\,, \quad \lambda_c &= 1.401155189\ldots \end{aligned} \tag{2}$$

We assume that the control parameter in (1) depends on n in accordance with $\Lambda_n = \exp(\mu + \nu\xi_n)$, i.e. we shall study the following equation:

$$x_{n+1} = \exp(\mu + \nu\xi_n)\frac{x_n}{\sqrt{1+x_n^2}}\,. \tag{3}$$

The values μ and ν may be considered as parameters controlling super-criticality and the intensity of the external force, respectively.

We believe that corresponding to our model system (2), (3) dynamics may be realized in physical experiments. For this purpose, one must take two systems — one exhibits Feigenbaum's period-doubling cascade, and the other demonstrates pitch-fork bifurcation — and introduce unidirectional coupling through nonlinear element having the sign-function characteristic.

3. Properties of the LSS

In this section we discuss in more detail a nature of time dependence of the external force [Procaccia *et al.*, 1987; Schroeder, 1991]. One can easily get the initial part of the LSS iterating numerically the Eq. (2) and writing down the signs of y_i ($i = 1, 2, 3, \ldots$):

$$\underline{1\,-1\;\;1}\;\;1\;\;\underline{1\,-1\;\;1}\,-1\;\;\underline{1\,-1\;\;1}\;\;1\;\;\underline{1\,-1\;\;1}\;\;1\;\;\underline{1\,-1\;\;1}\;\;1\;\;\underline{1\,-1\;\;1}\,-1$$

The underlined block $\underline{1\,-1\;\;1}$, that we call the basic block, is repeated many times, and the neighboring blocks are separated by a single term 1 or −1. Observe that by removing the underlined blocks, we obtain the sequence which exactly coincides with the initial one.

We can say that the LSS is determined uniquely by its basic block. Indeed, we can restore the entire sequence in the following way. Writing down the basic block, we get the first three terms of the sequence. Then we reproduce the first term at the fourth place and again write the basic block:

$$1\,-1\;\;1\;\;1\;\;1\,-1\;\;1\,.$$

Now we repeat the second term at the eighth place, and after that write the basic block again. It is clear that the procedure may be repeated *ad infinitum* and allows to find any number of terms for the LSS.

The described property may be interpreted as self-similarity: the LSS is invariant under the substitution

$$\begin{aligned} 1 &\to 1\,-1\;\;1\;\;1 \\ -1 &\to 1\,-1\;\;1\,-1. \end{aligned} \tag{4}$$

In other words, the initial sequence is obtained if we replace every term with the respective block from (4). Due to the self-similarity, we can consider the LSS as a kind of fractal signal [Kuznetsov *et al.*, 1991]. Evidently, the LSS has no finite period, but it is not random because there is a high degree of regularity in its structure.

4. Renormalization Group Analysis

Following the approach developed by Kuznetsov *et al.* [1995] let us represent the dynamical variable in Eq. (3) as $x_n = C_n z_n$ where z_n obeys the relation:

$$z_{n+1} = z_n \exp(\mu + \nu\xi_n)\,, \tag{5}$$

and $z_0 = 1$. Then, comparing (3) and (5), we obtain

$$C_{n+1} = \frac{C_n}{\sqrt{1+(C_n z_n)^2}}$$

or

$$\frac{1}{(C_{n+1})^2} = \frac{1}{(C_n)^2} + (z_n)^2\,. \tag{6}$$

Let us iterate (5) and (6) to construct the evolution operator determining dynamics over 4 time steps. (We have chosen the number 4 because it is the smallest period of approximate reproducing the LSS structure. It may be called the quasiperiod.) Obviously, we obtain:

$$\frac{1}{(C'_{n+4})^2} = \frac{1}{(C'_n)^2} + (z'_n)^2\alpha(\mu', \nu')^2\,, \tag{7}$$

$$z'_{n+4} = z'_n \exp(4\mu' + \nu'\sigma_n(4))\,, \tag{8}$$

where:

$$\begin{aligned} \alpha(\mu', \nu')^2 = 1 &+ \exp(2\mu' + 2\nu'\sigma_n(1)) \\ &+ \exp(4\mu' + 2\nu'\sigma_n(2)) \\ &+ \exp(6\mu' + 2\nu'\sigma_n(3))\,, \end{aligned} \tag{9}$$

and the function $\sigma_n(m)$ is a sum of m elements of the sequence ξ_i starting from the nth:

$$\sigma_n(m) = \begin{cases} \sum_{i=0}^{m-1} \xi_{n+i}\,, & \text{if } m > 0 \\ 0, & \text{if } m = 0\,. \end{cases} \tag{10}$$

Using the self-similarity of ξ_i, we replace its terms from n to $n+3$ by one term in accordance with (4) and rescale time $n/4 \to n$:

$$\frac{1}{(C'_{n+1})^2} = \frac{1}{(C'_n)^2} + (z'_n)^2\alpha(\mu', \nu')^2\,, \tag{11}$$

$$z'_{n+1} = z'_n \exp(4\mu' + \nu' + \nu'\xi_n)\,. \tag{12}$$

Noting the coincidence of the three first terms in both substitution blocks (4), we easily obtain the following relation for the factor α:

$$\alpha(\mu', \nu')^2 = 1 + \exp(2\mu' + 2\nu') + \exp(4\mu') + \exp(6\mu' + 2\nu')\,. \tag{13}$$

Observe that α does not depend on n. At last, we substitute C'_n and z'_n with C_n and z_n:

$$C'_n = \frac{C_n}{\alpha(\mu', \nu')}\,, \tag{14}$$

$$z'_n = z_n\,, \tag{15}$$

where

$$\mu = 4\mu' + \nu'\,, \tag{16}$$

$$\nu = \nu'\,. \tag{17}$$

As a result, the Eqs. (11) and (12) describing evolution over 4 time steps are reduced to the form of initial relations (5) and (6). So, this procedure may be repeated again and again many times given a sequence of evolution operators over 4^r time steps. Reproducing of a form for the evolution operators means that the dynamics of the system must demonstrate certain scaling properties. The relations (16) and (17) represent actually the RG transformation of the parameters.

The bifurcation transition in our model system (3) occurs at some critical line in the parameter plane (μ, ν). The critical situation may be found from a condition of existence of a fixed point for the RG transformation (16) and (17). Hence, the following relation defines the critical line:

$$3\mu + \nu = 0\,. \tag{18}$$

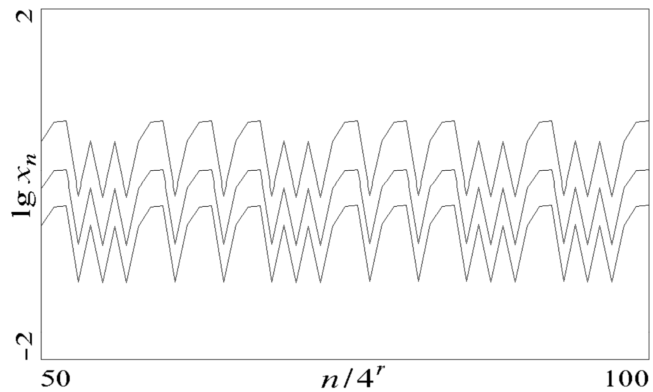

Fig. 1. Logarithm of the dynamical variable x_n of the system (2) and (3) versus rescaled time $n/4^r$ (from top to bottom: $r = 0, 1, 2$). The parameter $\nu = 1$ is constant for all curves and initial value (for $r = 0$) of μ is equal to 0.7. The other μ is rescaled according to (16). All graphs are seen to reproduce each other with a shift corresponding to the logarithm of the scaling factor α.

The scaling factor obeys the formula:

$$\alpha(\nu)^2 = 2 + 2\cosh\left(\frac{4\nu}{3}\right)\,. \tag{19}$$

In Fig. 1 we present the computer illustration for scaling property in the dynamics of our system (3). Logarithm of the dynamical variable x_n is plotted versus rescaled time $n/4^r$, for $r = 0, 1, 2$. The parameter ν is constant for all curves, and μ is rescaled according to (16). Observe remarkable correspondence of all the curves; the only difference is that they are shifted from each other by a distance given by the logarithm of the scaling factor α.

To calculate the critical exponent γ, we note that exactly at the critical point, increasing time by factor of 4 corresponds to decreasing the amplitude by factor of α [see (14)]. Thus, C_n must decay in time according to the power law $C_n \propto n^\gamma$ with the exponent:

$$\gamma(\nu) = -\frac{1}{2}\log_4\left(2 + 2\cosh\left(\frac{4\nu}{3}\right)\right)\,. \tag{20}$$

To check this conclusion numerically, we depict in Fig. 2 the plot of C_n versus time in a double logarithmic scale. Three presented curves relate to different ν, and values of μ are selected to correspond to critical points. Straight lines are drawn with slope given by (20). Observe that average slopes of the "experimental" curves coincide with the theoretical prediction.

In Fig. 3 a dependence of the critical exponent γ on the parameter ν is shown. At $\nu = 0$ the exponent is equal to the classical value: $\gamma(0) = -1/2$.

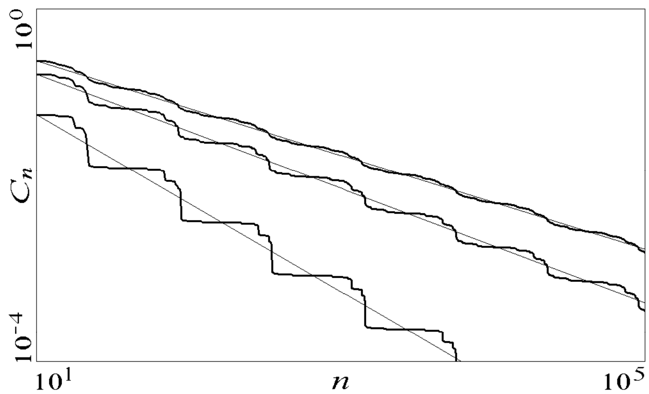

Fig. 2. The amplitude C_n of the system (2) and (3) versus time in double logarithmic scale. The graphs are plotted for critical points when $\nu = 0.5$, 1, 2 (from top to bottom). The straight lines are drawn for every ν with the slope $\gamma(\nu)$ given by (20). One can see that these lines describe well the critical decay of the "experimental" curves.

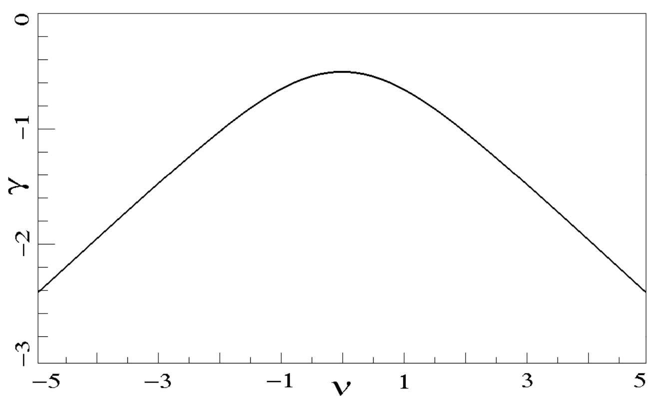

Fig. 3. The function $\gamma(\nu)$ (formula (20)). The critical exponent is equal to the classical value $-1/2$ when $\nu = 0$.

This case corresponds to pitch-fork bifurcation in the system (1) without external perturbation.

Now let us find another critical exponent that is responsible for the dependence of amplitude on the parameter of supercriticality near the critical point. It is convenient to designate $M = \exp(\mu)$. The critical situation corresponds to $M = M_c = \exp(-\nu/3)$. Let us suppose that the difference $(M - M_c)$ is small. The rule for transformation of $(M - M_c)$, when time is rescaled by factor of 4, is evident from (16):

$$M - M_c = 4(M' - M_c)\,. \tag{21}$$

Near the critical point we must have

$$C \propto (M - M_c)^{\beta}\,. \tag{22}$$

Due to rescaling rules (14) and (21) we find

$$\beta(\nu) = \frac{1}{2}\log_4\left(2 + 2\cosh\left(\frac{4\nu}{3}\right)\right)\,. \tag{23}$$

Note that $\beta = -\gamma$ as well as in the classical case.

Usually, RG approach gives asymptotic relations that are valid in narrow neighborhood of a critical point. It is interesting to note that in our particular system the obtained quantitative relations for parameters are correct for arbitrary values of deflection from the criticality.

5. More General Class of Fractal Sequences

We have described in Sec. 3 how to restore the LSS using its basic block. It is clear that we can generalize this procedure using the same algorithm with other basic blocks. For instance, having chosen the basic block $\underline{1\ {-1}\ 1\ 1}$, we get the sequence:

$$\underline{1\ {-1}\ 1\ 1}\ \ 1\ \ \underline{1\ {-1}\ 1\ 1}\,{-1}\ \ \underline{1\ {-1}\ 1\ 1}\ \ 1\ \ \underline{1\ {-1}\ 1\ 1}\ \ 1\ \ \underline{1\ {-1}\ 1\ 1}\ \ 1$$

So, we can consider the class of self-similar sequences. Moreover, we may generalize the results obtained above to the case when the system (3) is influenced by any sequence from this class.

Let us consider a sequence with the quasiperiod m. Then using RG approach, we have to consider an evolution operator over m time steps and try to reduce it to a form of the initial map by variable and parameter changes. As a result, the following generalized relations may be obtained that are analogous to (14)–(17):

$$\mu = m\mu' + \sigma_0(m-1)\nu'\,, \tag{24}$$

$$\nu = \nu'\,, \tag{25}$$

$$C'_n = \frac{C_n}{\alpha(\mu', \nu')}\,, \tag{26}$$

$$z'_n = z_n\,, \tag{27}$$

where:

$$\alpha(\mu, \nu)^2 = \sum_{i=0}^{m-1}\exp(2\mu i + 2\nu\sigma_0(i))\,. \tag{28}$$

The fixed point of (24) and (25) defines the critical line in the parameter plane (μ, ν):

$$\mu + \frac{\sigma_0(m-1)}{m-1}\nu = 0\,, \tag{29}$$

where transition of the system from zero to nonzero amplitude state occurs.

To find the respective critical exponents, we repeat the previous considerations and obtain

$$\gamma(\nu) = -\frac{1}{2}\log_m\left(\sum_{i=0}^{m-1}\exp\left(-2\frac{\sigma_0(m-1)}{m-1}\nu i + 2\nu\sigma_0(i)\right)\right), \tag{30}$$

$$\beta(\nu) = -\gamma(\nu)\,. \tag{31}$$

The scaling properties of the generalized system are illustrated in Figs. 4 and 5. The basic block 1 −1 1 1 is taken as an example; the corresponding quasiperiod is $m = 5$. In Fig. 4 we show the logarithm of the dynamical variable x_n versus rescaled time $n/5^r$ for three steps of renormalization ($r = 0$, 1, 2). The curves are similar, the shift is given by the logarithm of the scaling factor.

Critical decay of the amplitude C_n is demonstrated in Fig. 5. Double logarithmic scale is used here. The curves are in a good agreement with straight lines of a slope obtained from (30). Critical exponent as a function of ν is presented in Fig. 6. The exponent takes the classical value $-1/2$ if $\nu = 0$. Note asymmetry of this curve, unlike the graph in Fig. 3.

In conclusion, we have studied bifurcation in the system (3) that is under the influence of the class of self-similar sequences. We find that the situation is characterized by nonclassical critical exponents. These exponents appear to depend on the intensity of the parameter modulation. Presence of nonclassical critical exponents make us conjecture that the observing bifurcation gives rise to the appearance of a kind of strange nonchaotic attractor [Grebogi *et al.*, 1984]. (It may be checked by direct computer calculations that the Liapunov exponents for the system (3) driven by the external force (2) are nonpositive in the supercritical field, and the

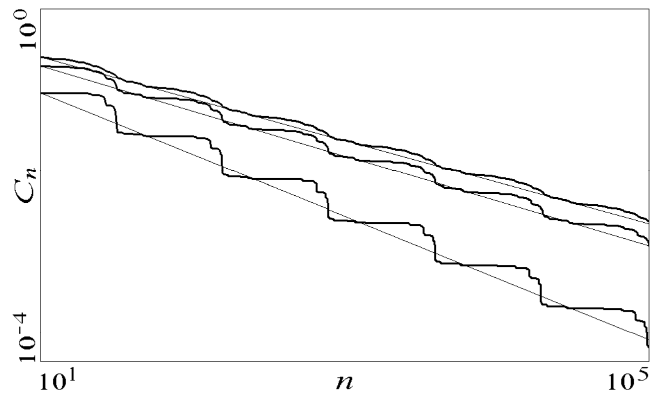

Fig. 5. The amplitude C_n of the system (3), when the influence sequence has basic block 1 −1 1 1, against time at critical points for $\nu = 0.5$, 1, 2 (from top to bottom). The graphs are plotted in double logarithmic scale. The straight lines have the slope $\gamma(\nu)$ given by Eq. (30) and are plotted for every value of ν.

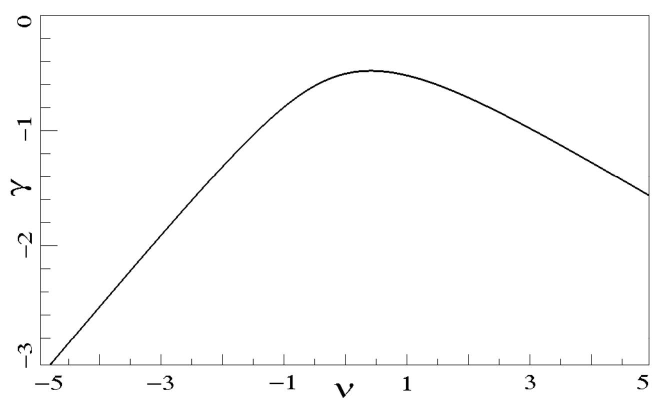

Fig. 6. The function $\gamma(\nu)$ when the sequence with basic block 1 −1 1 1 influences the system (3) [Eq. (30)]. If $\nu = 0$ the critical exponent takes the classical value $\gamma(0) = -1/2$. Note that this curve is not symmetric despite the graph in Fig. 3.

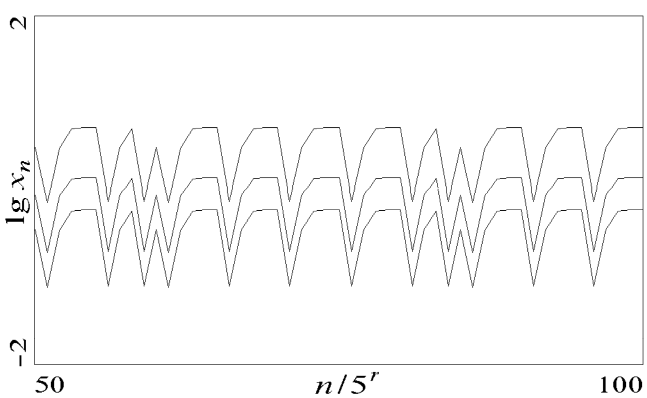

Fig. 4. Logarithm of the dynamical variable x_n versus rescaled time $n/5^r$ for $r = 0$, 1, 2 when the system (3) is under influence of the sequence with basic block 1 −1 1 1. The upper line corresponds to $r = 0$ and is plotted for $\mu = 0.7$, $\nu = 1$. The parameters of the other curves are calculated in accordance with (24) and (25). The shift between these same curves is equal to the logarithm of the scaling factor α.

attractor of this system has a fractal structure. This structure is not subordinate strongly to the external force, because its scaling characteristics depend on the intensity of coupling.) Previously known examples were connected to quasiperiodic perturbations with a definite irrational frequency. Here we deal with another class of temporal dependence for the external force. The simplest representative of this class is LSS, the sequence of signs generated by the logistic map at the accumulation point of period-doubling cascade. For this case, it seems possible to observe the transition in experiments.

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