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# GENERALIZED DIMENSIONS OF FEIGENBAUM'S ATTRACTOR FROM RENORMALIZATION-GROUP FUNCTIONAL EQUATIONS

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A method is suggested for the computation of the generalized dimensions of fractal attractors at the period-doubling transition to chaos. The approach is based on an eigenvalue problem formulated in terms of functional equations with coefficients expressed in terms of Feigenbaum's universal fixed-point function. The accuracy of the results depends only on the accuracy of the representation of the universal function.

The multifractal or thermodynamic formalism is an important tool for description of strange sets arising in dynamical systems in different contexts. Its basic ideas have been clearly formulated by, for example, Halsey et al. [1]. Some of the examples presented by these and other authors are related to the fractal attractors that occur at the onset of chaos via period doubling and quasiperiodicity [2]–[8]. The multifractal analysis reveals global scaling properties of these attractors, such as the generalized dimensions and the $f(\alpha)$ spectra. They are of principal interest because of their universality for systems of different nature. Moreover, their characteristics can be determined through experiments [8].

One of the well-studied multifractal objects is the Feigenbaum attractor, which occurs at the period-doubling transition to chaos in unimodal one-dimensional maps with quadratic extremum and in a wide class of more general nonlinear dissipative systems [9, 2, 10]. Beside the original procedure of Halsey et al. (namely the construction and analysis of the partition functions defined as sums over some natural covering of the attractor), several other approaches to the computation of the multifractal characteristics have been developed. Bensimon et al. [3] used a method based on a break up of a partition sum into two components with subsequent use of the scaling property. Aurell revisited this method and noted its link with Feigenbaum's trajectory scaling function [4]. Kovács [5] suggested a procedure of extracting the dimensions from the eigenvalue problem for the Frobenius–Perron operator. Christiansen et al. [6] used the idea of approximating the strange sets by periodic orbits and expressed the desired quantities in terms of cycle expansions. (To our knowledge, the estimate of the Hausdorff dimension of the Feigenbaum attractor in Ref. [6] remains the most precise to date.)

In some sense, the *global* description of scaling properties in the multifractal formalism seems opposite to the *local* description in terms of the Feigenbaum renormalization group approach [9]. The latter is based on the solution of the functional fixed-point equation and associated with scaling relations for the evolution operators in a neighborhood of the extremum of the map under consideration.

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In this paper we present a novel method for precise computation of the multifractal characteristics: the problem may be presented in terms of the Feigenbaum renormalization transformation applied to some auxiliary function. The desired quantities, such as the generalized dimensions and the $f(\alpha)$ spectrum, can be obtained from an eigenvalue problem represented as a functional equation involving Feigenbaum's universal fixed-point function. An analogous approach was previously suggested in application to the problem of the influence of noise on the period-doubling transition [11, 2]. That problem appears to be linked with one special generalized dimension, as noted e.g. in [2, 12], and this speaks in support of the generalization we undertake here. A similar idea was used in Ref. [13] for study of scaling regularities in the Fourier spectrum and response function of a quadratic map at the period-doubling transition to chaos.

Using the representation of the Feigenbaum function from Ref. [14] we calculate the generalized dimensions with high precision; these values are in excellent agreement with the previously obtained results. Also we present here results related to the generalized dimensions of the multifractal attractors at the onset of chaos in the unimodal maps of degrees 4, 6, and 8, for which the universal function $g(x)$ is available in form of numerically found polynomial expansion [15].

To estimate the multifractal characteristics of the Feigenbaum attractor by use of the standard approach, the generalized partition functions $\Gamma_k(q,\tau)=\sum\limits_{i=1}^{2^k} p_i^q/l_i^\tau$ are employed. Here $q$ and $\tau$ are real parameters, $p_i=2^{-k}$, $l_i=|x_i-x_{i+2^k}|$, and the sequence $x_i$ results from iterations of the unimodal map at the limit point of the period-doubling accumulation starting at the extremum point. Obviously, $\Gamma_k(q,\tau)=2^{-qk}S_{2^k}(\tau)$, where $S_{2^k}(\tau)=\sum\limits_{i=1}^{2^k} l_i^{-\tau}$. For each given $q$ an appropriate value $\tau=\tau(q)$ may be found such that $\Gamma_{k+1}(q,\tau)=\Gamma_k(q,\tau)$ (as $k\to\infty$). Vice versa, for a given $\tau$ we can choose of $q=q(\tau)$. This relation between $q$ and $\tau$ is used below to obtain the generalized dimensions and $f(\alpha)$ spectrum.

For large $k$ the lengths of the intervals $l_i$ are small, and they can be estimated via the derivatives as

$$l_i \cong |dx_i/dx_1| l_1. \tag{1}$$

In this approximation, we can compute them step by step together with the sums $S$ via simultaneous iterations of the relations

$$\begin{aligned} x_{i+1} &= f(x_i),\\ l_{i+1} &= |f'(x_i)| l_i,\\ S_{i+1} &= S_i + l_{i+1}^{-\tau}\Psi(x_i). \end{aligned} \tag{2}$$

For the moment, the auxiliary function $\Psi(x)$ is supposed to be identically equal to 1.

From (2), we obtain

$$\begin{aligned} x_{i+2} &= f(f(x_i)),\\ l_{i+2} &= |f'(f(x_i))f'(x_i)| l_i,\\ S_{i+2} &= S_i + l_{i+2}^{-\tau}\left[|f'(f(x_i))|^\tau \Psi(x_i) + \Psi(f(x_i))\right]. \end{aligned} \tag{3}$$

By performing Feigenbaum's scale change $x\mapsto x/\alpha$, $l\mapsto l/|\alpha|$ ($\alpha$ is the Feigenbaum constant), we get the equations identical in form to (2) but with new functions $f_{\text{new}}(x)=\alpha f(f(x/\alpha))$, $\Psi_{\text{new}}(x)=$ $=L_{f,\tau}\Psi(x)$, where $L_{f,\tau}$ is the linear operator

$$L_{f,\tau}:\Psi(x)\mapsto |\alpha|^\tau\left[|f'(f(x/\alpha))|^\tau \Psi(x/\alpha)+\Psi(f(x/\alpha))\right]. \tag{4}$$

This transformation may be repeated again and again. Asymptotically, $f(x)$ converges to the fixed-point function satisfying the Feigenbaum–Cvitanović equation $g(x)=\alpha g(g(x/\alpha))$, and the sequence $\Psi_k(x)$ will follow the recursion $\Psi_{k+1}(x)=L_{g,\tau}\Psi_k(x)$.

Thus, as $k \to \infty$, the function $\Psi_k$ tends to the eigenfunction associated with the largest eigenvalue of the linear operator $L_{g,\tau}$, the eigenproblem being

$$\nu(\tau)\Psi(x) = |\alpha|^\tau \left[ |g'(g(x/\alpha))|^\tau \Psi(x/\alpha) + \Psi(g(x/\alpha)) \right]. \tag{5}$$

(A particular case of this equation for $\tau = 2$ appears in the theory of effect of noise on the period-doubling transition. A possibility of computation of the noise scaling constant via sums of the derivatives over the Feigenbaum attractor was noted in, for example, [16, 12].)

Thus, we see that the eigenvalue $\nu(\tau)$ indicates the rate of growth or decrease of the sums $S$:

$$S_{2^k}(\tau) \propto \nu(\tau)^k. \tag{6}$$

To obtain $\Gamma_k \to \text{const}$ as $k \to \infty$, we must have

$$\nu(\tau) = 2^q, \text{ or } q = \log_2 \nu(\tau). \tag{7}$$

Then, in accordance with the multifractal formalism, we can obtain the generalized dimensions

$$D_q = \frac{\tau}{q-1}, \tag{8}$$

and the $f(\alpha)$ spectrum as an implicitly defined relation between the variables

$$\alpha = \frac{d\tau}{dq} \text{ and } f = q\frac{d\tau}{dq} - \tau. \tag{9}$$

Although our argumentation starts from the approximate relation (1), we believe that the final Eq. (5) is exact. The numerically obtained data presented below strongly supports this conjecture: the generalized dimensions are in excellent agreement with the best known numerical results, up to all reliable digits. Apparently, for large $k$ (1) is indeed a good approximation. One might hope that a rigorous proof can be found.

We have performed numerical analysis of the eigenvalue problem (5) for the classic Feigenbaum attractor of the quadratic map and for unimodal maps of even integer degrees $d = 4$, 6, and 8. In principle, the accuracy of the results is determined only by the accuracy of the approximation of the universal functions. Moreover, these accuracies are of the same order. (This is in contrast with the approach of [3, 4]: we do not use iterative calculations in which the universal function $g(x)$ is engaged. Such calculations inevitably lead to decrease of the accuracy.)

With the known polynomial approximation of $g(x)$ and the value of the scaling constant $\alpha$ we have numerically performed the functional transformation defined by the right-hand side of Eq. (5). The unknown function $\Psi(x)$ is represented by its values at the nodes of a one-dimension grid on the interval $[0, 1]$ and by an interpolation scheme between the nodes. In actual computations it was convenient to use a grid of constant step along the axis of variable $y = |x|^d$ and a fourth-order interpolation in terms of $y$. Given the input values of $\Psi(x)$ the program yields an analogous table as output.

Suppose we fix $\tau$ and wish to estimate $q$. For the initial condition $\Psi(x) \equiv 1$, we perform the functional transformation, and normalize the resulting function as $\Psi^0(x) = \Psi(x)/\Psi(0)$. The new function is taken as the initial condition and so on. This operation is repeated many times until the form of the function $\Psi(x)$ stabilizes. Then, the value of $\Psi(0)$ before the normalization is taken to be $\nu(\tau)$, and we finally set $q(\tau) = \log_2 \nu(\tau)$.

To find $\tau$ for a given $q$ we use the above procedure together with a simple iteration scheme for the numerical solution of the algebraic equation $q(\tau) = q$. We may then calculate $D_q = \tau/(q-1)$ at $q \neq 1$. In particular, $D_0$ is the Hausdorff dimension, and $D_2$ is the correlation dimension.

To obtain the information dimension $D_1$ it is necessary to calculate the limit as $q \to 1$, that is, as $\tau \to 0$. Formally, this follows from L'Hôpital's rule: $D_1 = \lim\limits_{q\to 1} \dfrac{\tau(q)}{q-1} = \left(\dfrac{d\tau}{dq}\right)_{q=1} = \left(\dfrac{dq}{d\tau}\right)^{-1}_{\tau=0}$. To compute this without sacrifice of precision we use the following algorithm. For $\tau \ll 1$ let us write $\Psi_k(x) = 2^k|\alpha|^{k\tau}[1+\tau h_k(x)]$ and substitute this into Eq. (5). Retaining only first-order terms, we get

$$h_{k+1}(x) = \tfrac{1}{2}\left[h_k(x/\alpha) + h_k(g(x/\alpha))\right] + \tfrac{1}{2}\ln|g'(g(x/\alpha))|\,. \tag{10}$$

Numerically, representing $h_k(x)$ by a table of its values and performing a large number of steps of the transformation, one can observe that $h_{k+1}(x) - h_k(x) \to \theta = \text{const}$ as $k \to \infty$. This implies that $\Psi_k \propto |\alpha|^{k\tau}2^k e^{k\theta\tau}$. On the other hand, $\Psi_k \propto 2^{kq(\tau)} \cong 2^{k(q+\tau dq/d\tau)} = 2^{k(1+\tau/D_1)}$. Hence,

$$D_1 = \frac{\ln 2}{\ln|\alpha| + \theta}. \tag{11}$$

For quadratic maps we have performed the computations based on a polynomial representation of $g(x)$ with coefficients taken from the paper by Lanford [14]. His data are of very high precision, but in our calculations the accuracy is limited due to the use of standard double-precision arithmetic. As a result, we get not more than 14 true digits in the decimal representation of the generalized dimensions. These data are presented in the first column of Table 1. Note excellent agreement of the Hausdorff dimension (up to the last decimal digit!) with the result of Christiansen et al. [6]. Other dimensions for the Feigenbaum attractor were presented by Kovács [5], and they coincide with our results up to the 10-th digit, the accuracy achieved in that work.

Table 1. Generalized dimensions for fractal attractors at the onset of chaos in unimodal maps of degree $d$

| | $d=2$ | $d=4$ | $d=6$ | $d=8$ |
|---|---|---|---|---|
| $D_5$ | 0.45392270234470 | 0.407695571 | 0.373232166 | 0.351475400 |
| $D_4$ | 0.46615155691823 | 0.426832904 | 0.392400635 | 0.370142909 |
| $D_3$ | 0.48077684940009 | 0.454569793 | 0.421638052 | 0.399231039 |
| $D_2$ | 0.49783645928917 | 0.495316676 | 0.468035066 | 0.447019466 |
| $D_1$ | 0.51709757255124 | 0.555181822 | 0.544847134 | 0.531111008 |
| $D_0$ | 0.53804514358055 | 0.642575065 | 0.683433256 | 0.707102082 |
| $D_{-1}$ | 0.55991291016494 | 0.763919555 | 0.946229117 | 1.146118382 |
| $D_{-2}$ | 0.58173600034603 | 0.894257449 | 1.205507002 | 1.510079742 |
| $D_{-3}$ | 0.60247817187829 | 0.992066238 | 1.354808070 | 1.698747772 |
| $D_{-4}$ | 0.62126594260209 | 1.056616863 | 1.445090859 | 1.811996998 |
| $D_{-5}$ | 0.63760518368338 | 1.100453275 | 1.505301852 | 1.887496869 |

In the remaining three columns of Table 1 we present results for the generalized dimensions of the multifractal attractors at the onset of chaos in unimodal maps of degrees 4, 6, and 8 obtained using the universal functions given in, for example, Ref.[15].

As an alternative to the traditional definition of the generalized dimensions $D_q$ one might consider a family of dimensions, $D^\tau$, namely, $D^\tau = D_{q(\tau)} = \tau/(q(\tau)-1)$. As mentioned above, for $\tau = 2$ the equation (5) is of the form studied in the theory of noise effect on the period-doubling transition [11]; the noise scaling constant is defined as $\gamma = \sqrt{\nu(2)}$. Hence, the dimension $D^2$ is related to the effect of noise. The scaling factor $\gamma$ is expressed via the dimension $D^2$ as $\gamma = 2^{1/D^2+1/2}$. In Table 2 we present high-precision data for values of $q(2)$, dimensions $D^2$, and noise scaling factors obtained from the numerical solution of the eigenproblem (5) for maps of degree 2, 4, 6, and 8. The factors $\gamma$ are calculated more accurately than ever before [17]. (Note that the value of $q$ depends on the degree $d$, so there does not exist a definite dimension from the family $D_q$ associated with the noise effect!)

For the problem of unimodal maps, our method of calculation of the generalized dimensions has no obvious computational advantages over those of [5, 6], but it represents the problem in a new perspective and indicates novel relations between global and local descriptions of scaling regularities.

Table 2. Generalized dimensions $D^2$ and noise scaling factors for unimodal maps of degree $d$

| | $d=2$ | $d=4$ | $d=6$ | $d=8$ |
|---|---|---|---|---|
| $q(2)$ | 5.45324245756108 | 6.086657808 | 6.654767241 | 7.070578662 |
| $D^2 = D_{q(2)}$ | 0.44911096107158 | 0.393185481 | 0.353683877 | 0.329457884 |
| $\gamma$ | 6.61903651081803 | 8.243910853 | 10.037886410 | 11.59386214 |

The renormalization-group equations can be similarly used for the calculation of multifractal properties in many other situations at the onset of chaos, e.g. in bimodal one-dimensional maps [18], asymmetric one-dimensional maps [19], two-dimensional period-doubling maps [20], quasiperiodically forced maps [21], and complex analytical maps [22]. Such an approach will be useful especially for situations when computations based on the traditional partition-function approach are difficult.

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