Scale-invariant structure of parameter space for coupled Feigenbaum systems

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1. The study of chaotic oscillations in dynamical systems has led recently to the development of several universal models for describing the onset of chaotic behavior.[1] One of these models (here called the Feigenbaum model) is a generalized nonlinear dissipative system with the property that as a certain parameter λ increases, a transition to chaotic behavior occurs via a hierarchy of period-doubling bifurcations, and the behavior becomes completely chaotic at a critical point λ_c (Ref. 2). The parameter space (the λ axis) has a scale-invariant structure near λ_c – the pattern that describes the regions of different behavior remains the same if $\lambda - \lambda_c$ is divided by $\delta = 4.669$. Examples of Feigenbaum systems include nonlinear dissipative oscillators driven by a periodic external signal,[3] Josephson junctions in an rf field,[4] radio-wave oscillators with inertial nonlinearity (Ref. 5), etc. The simplest example is the model[2]

$$x_{n+1} = \lambda - x_n^2, \tag{1}$$

where the variable x_n describes the state of the system at time n.

A natural area for the further development of the theory would be to use Feigenbaum systems to construct more complicated objects and study how chaos develops in them. For example, Refs. 6–8 numerically analyzed a system of two coupled systems governed by equations of the form (1). However, the properties of universality and similarity were not displayed, even though they are present for the individual oscillators. We will see that this is because two-parameter families of systems were analyzed; to get universal behavior, one must analyze three-parameter families.

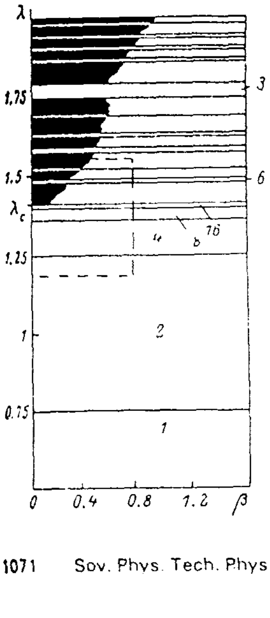

FIG. 1. The β, λ plane for a system with pure B coupling. The numbers labeling the stable regions of in-phase cycles give their periods. The hatching shows regions of in-phase chaotic oscillations, while the black areas correspond to out-of-phase oscillations. The same configuration is reproduced to smaller scale inside the rectangle bounded by the dashed lines.

It was shown in Ref. 9 that as far as the long-term behavior of coupled Feigenbaum systems is concerned, a weak coupling introduced in some arbitrary way is completely determined by two parameters α and β, which in the terminology adopted there correspond to two types of coupling A and B. These couplings transform differently under the renormalization group[2]: the scaling factor is $a = -2.503$ in one case and $b = 2$ in the other. The results in Ref. 9 imply that the parameter space (α, β, γ) for coupled systems has a scale-invariant structure near the point (0, 0, λ_c) – the points (α, β, $\lambda_c + \Lambda$) and (α/a, β/b, $\lambda_c + \Lambda/\delta$) correspond to similar system behavior with time scales that differ by a factor of 2.

In this note we discuss numerical calculations which yield specific information about the structure of parameter space for coupled Feigenbaum systems and demonstrate the validity of the above similarity law.

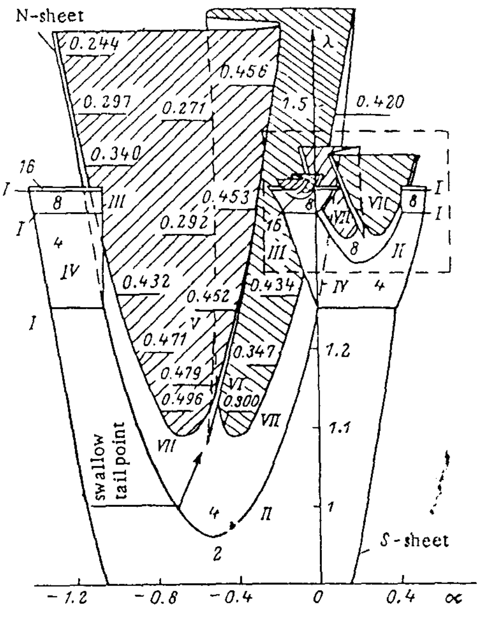

FIG. 2. The surface (α, λ) for a system with pure A coupling. The S-sheet corresponds to in-phase motion of the coupled systems, while the N-sheets are out-of-phase. The unhatched regions on the S- and N-sheets describe stable cycles with the indicated periods. The bifuration lines are labeled by Roman numerals: I) period-doubling lines for in-phase cycles; II) period-doubling lines with soft generation of a nonsynchronous cycle (5) along which the S- and N-sheets are attached; III) edges of the S-sheet at which a jump occurs to an N-sheet; IV) edges of N-sheets at which a jump to the S-sheet takes place; V, VI) fold lines on the N-sheets; VII) Hopf bifurcation line (the rotation numbers of the associated quasiperiodic attractor are indicated alongside). The configuration repeats on a smaller scale inside the dashed rectangle (the orientation relative to the horizontal axis also changes).

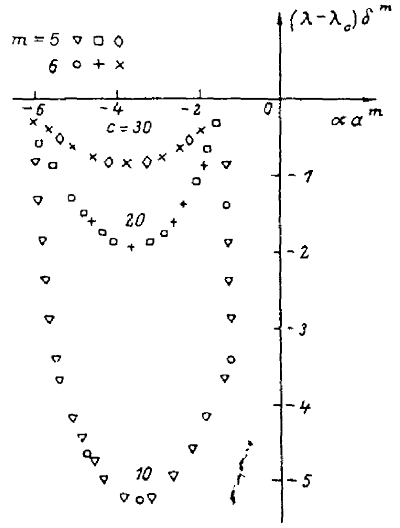

FIG. 3. Boundaries on the stable regions for in-phase 2^m-cycles on the surface (α, β, λ), where $\beta = c|\lambda - \lambda_c| \log \delta^b, \lambda)$.

2. We will analyze the following system of two identical symmetrically coupled systems governed by equations of the form (1):

$$x_{n+1} = \lambda - x_n^2 + \varepsilon(x_n - y_n) + \mu(x_n^2 - y_n^2),$$
$$y_{n+1} = \lambda - y_n^2 + \varepsilon(y_n - x_n) + \mu(y_n^2 - x_n^2), \qquad (2)$$

where the parameters ε and μ specify the coupling. In terms of the variables $\xi_n = (x_n + y_n)/2$ and $\eta_n = (x_n - y_n)/2$ Eqs. (2) become

$$\xi_{n+1} = \lambda - \xi_n^2 - \eta_n^2, \quad \eta_{n+1} = -2B(\xi_n + \alpha)\eta_n, \qquad (3)$$

where $B = 1 - 2\mu$ and $\alpha = \varepsilon/(1 - 2\mu)$. Rather than considering the set (α, B, λ) we will use the set (α, β, λ), where β is defined by

$$B = e^{-\beta}(1 - 0.6025\alpha + 0.1019\alpha^2 - 0.0278\alpha\lambda)^{0.2779}. \qquad (4)$$

The numerical constants in Eq. (4) have been chosen so that α and β agree with their values for type-A and type-B coupling.[9] We will first examine pure A and pure B coupling.

3. For pure B coupling we have $\alpha = 0, \beta > 0$.[1)] In this case both of the coupled systems oscillate stably and in-phase if $\lambda < \lambda_c$ ($x_n \equiv y_n, \eta_n \equiv 0$). The oscillation period doubles at the same values of λ as for an isolated Feigenbaum system (1).

For $\lambda > \lambda_c$ the in-phase motion may become unstable if the Lyapunov characteristic exponent[10] $\gamma(\lambda) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|2x_n|$ of the attractor for (1) exceeds β (Ref. 11). Thus for $\lambda > \lambda_c$, regions of in-phase $[0 < \gamma(\lambda) < \beta]$ and out-of-phase $[\gamma(\lambda) > \beta)$ chaotic oscillations are separated by islands of periodic in-phase oscillations $[\gamma(\lambda) \equiv 0$, "windows of stability"[2]]. Figure 1 shows the configuration of these regions in the β, λ plane; note that it remains unchanged if β and $\lambda = \lambda_c + \Lambda$ are replaced by β/b and $\lambda_c + \Lambda/\delta$.

4. Pure A-coupling corresponds to $\beta = 0$, $\alpha \neq 0$. It is helpful to regard the parameter surface (α, γ) as composed of sheets, one of which (the S-sheet) corresponds to in-phase motion of the subsystems, while the others (N-sheets) describe nonsynchronous motion. The sheets are joined along curves, and crossing from one sheet to another across these curves corresponds to a soft transition between in-phase and out-of-phase oscillations. The edges of the sheets which are not joined exhibit "hard jump bifurcation" from one sheet to another, which is accompanied by hysteresis. Figure 2 shows the numerically calculated configuration of regions on the (α, λ) surface. The structural similarity is clearly evident-the configuration in Fig. 2 remains the same if α is replaced by α/a and $\lambda = \lambda_c + \Lambda$ by $\lambda_c + \Lambda/\delta$.

When the line joining the S and N sheets is crossed from below, a nonsynchronous cycle is generated in which the phases of the motion of the subsystems differ by a half-period,

$$x_{n+M/2} = y_n, \quad y_{n+M/2} = x_n \quad \text{or} \quad \xi_{n+M/2} = \xi_n, \quad \eta_{n+M/2} = -\eta_n. \qquad (5)$$

Figure 2 indicates the period M for several of the N-sheets.

Each N-sheet contains a swallowtail point[12] which is approached by the fold lines denoted by V and VI in Fig. 2. As we move clockwise around the swallowtail point, the attractor [cycle (5)] evolves continuously until line VI is crossed, after which the nature of the oscillations changes abruptly and the newly formed cycle is again described by (5). For a counterclockwise circuit, the jump occurs when line V is crossed.

The hatching in Fig. 2 shows where cycle (5) is unstable. When the boundaries of the hatched regions are crossed, a Hopf bifurcation occurs and a quasiperiodic attractor forms. The rotation number w (i.e., the ratio of the periods of the old and new cycles) is indicated alongside the bifurcation line. The hatched regions have a fine structure in which "fingers" of synchronous motion extend out to points on the bifurcation line for which $w = p/q$ is rational. Inside each finger the quasiperiodic oscillations are replaced by a complex cycle of period Mq. Chaotic behavior develops as we move deeper into the hatched regions (cf. also Refs. 6 and 8).

5. Finally, consider the case of mixed coupling, $\alpha \neq 0$, $\beta < 0$. Since it is difficult to sketch the structure of the three-dimensional parameter space, we will make do with a brief qualitative description and quantitatively verify the similarity relations.

Imagine that the B-coupling is gradually "turned on in a system with pure A coupling. Calculations of the evolution of the regions on the (α, λ) surface show that for small β the surface continues to be composed of S- and N-sheets attached along certain curves, and the overall position of the regions on the sheets remains unchanged. As β increases, fingers of nonsynchronous oscillations grow upward and terminate in regions where $\lambda < \lambda_c$. In order for each subsequent finger to disappear the parameter β must be doubled.

We will verify the similarity law by examining the cross section of α, β, λ parameter space defined by the surface $\beta = c|\lambda - \lambda_c| \log \delta^b$, where c is an arbitrary constant. The configuration of the regions on this surface should remain the same if β and $\lambda - \lambda_c$ are divided by α and δ, respectively (in this case α changes by a factor of b). Figure 3 plots the stability boundary curves for in-phase cycles of period 2^m in αa^m versus $(\lambda - \lambda_c)\delta^m$ coordinates for several values of m and c. We see that the points lie on the same curves as m varies.

Thus if we follow a path in (α, β, λ) space along which λ increases, various types of chaotic behavior will be encountered, depending on the path: 1) there may be in infinite sequence of period doublings; 2) a finite number of doublings may be followed by onset of quasiperiodic oscillations which then break down; 3) an abrupt transition to out-of-phase chaos may occur after finitely many doublings. The scale invariance cannot be detected by studying individual paths only — instead, a global analysis of (α, β, λ) parameter space is required.

[1)] Equations (3), (4) show that the B coupling tends to make the states of the individual subsystems approach one another if and only if $\beta > 0$; $\beta < 0$ can occur only if amplification is present in the coupling channel, which we do not consider here.

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