Physics Letters A 184 (1994) 413–421
North-Holland

PHYSICS LETTERS A

From bimodal one-dimensional maps to Hénon-like two-dimensional maps: does quantitative universality survive?

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Received 19 August 1993; accepted for publication 23 November 1993
Communicated by A.R. Bishop

It is shown that some types of quantitative universality occurring at the transition to chaos in two-parameter families of bimodal one-dimensional maps can be observed also in two-parameter families of two-dimensional dissipative Hénon-like maps.

1. Transition to chaos via period doubling cascade is widespread in the world of nonlinear dissipative systems. It is known that this scenario is characterized by remarkable properties of universality and scaling discovered by Feigenbaum using a renormalization group (RG) analysis of unimodal (i.e., having one extremum) one-dimensional maps [1,2].

One of the promising extensions of Feigenbaum's approach is a theory of period doublings in two-parameter families of bimodal (i.e., having two extrema) one-dimensional maps [3–9]. It was found that the boundary of chaos in a bimodal map parameter plane contains both Feigenbaum's lines of period doubling accumulation and a Cantor-like set of points with special properties. The points of the latter set will be further referred to as *critical points of codimension two*. The simplest representative of them is a *tricritical point* [10]. In the parameter plane tricritical points appear as terminal points of Feigenbaum's critical lines.

In this paper we want to discuss the following question: Does the boundary of chaos in more common nonlinear systems posses the structure characteristic for bimodal one-dimensional maps? As an example we consider a class of Hénon-like maps

$$F\colon (x, y) \to (f(x) - Dy, x)\ , \qquad (1)$$

where the function $f(x)$ depends on the two parameters A and B and has two extremal points, a maximum and a minimum. The parameter D is responsible for engaging the second dimension: when $D=0$ eq. (1) is reduced to a one-dimensional bimodal map.

Due to the know result of Collet, Eckmann and Koch [2] we may be sure that the Feigenbaum segments of the chaos boundary continue to exist when $D \neq 0$. On the other hand, it was shown that tricritical points do not survive when the second dimension is engaged [11]. In typical families of two-dimensional maps tricriticality appears only as a phenomenon of a higher codimension three. This result seems dramatic if we consider the possibility of theory extension for multidimensional systems. However, in the present paper we reach a more optimistic conclusion. On the basis of numeric results and some heuristic arguments we assert that other types of codimension-two criticality survive in Hénon-like maps. As far as we know, for the non-one-dimensional case this is the first report on the quantitative universality occurring in bimodal one-dimensional maps.

2. After the works of Kapral, MacKay and other investigators [3–9] it is known that codimension-two critical points at the boundary of chaos are accumulation points of untypical period doubling cascades observed along certain particular paths in a parameter plane. A set of this trajectories forms a binary tree, each of the critical points being coded with an infinite sequence of symbols U and D (or, in an al-

SSDI 0375-9601(93)E0950-F

ternative designation, L and R [4–8]).

The simplest way to find numerically the coordinates of a codimension-two critical point for a one-dimensional map is to find the accumulation point of the appropriate sequence of double superstable cycles. These cycles are defined as cycles having both extremal points as their elements. A double superstable cycle will be referred to as (p, q) cycle if the maximum is mapped to the minimum after p iterations, and the minimum is mapped to the maximum after q iterations.

To find the (p_i, q_i) cycle sequence for a definite UD code we use the following rule. We start with a $(p_1=1,\ q_1=1)$ cycle. (The corresponding point in the parameter plane is usually easy to find.) Then taking consequently the symbols from the UD code we calculate p and q,

$$(p_{i+1}, q_{i+1}) = (p_i, p_i+2q_i), \quad \text{if U}, \\ = (2p_i+q, q_i), \quad \text{if D}. \tag{2}$$

Note that the periods of the cycles appear to be equal to $p_i+q_i=2^i$.

Each arbitrary infinite UD code generates its own codimension-two critical point. It means that these points form a set of the continuum power. However, in this paper we restrict our consideration to a smaller class of points having UD codes with periodic tails.

Suppose we take a definite UD code with a tail formed by a multiple repetition of a k-symbol segment, and then we find the generated critical point. At this point the map describing the dynamics over a long time interval of $(2^k)^n$ steps is defined (depending on the n normalization condition) by a universal function which is determined by the tail type only. This function $g(x)$ appears to be a solution of the k-fold iterated Feigenbaum–Cvitanovich RG equation,

$$g(x) = a g^{2^k}(x/a), \tag{3}$$

where a is a factor which has to be found together with the solution of (3).

The scaling properties in the vicinity of the critical point in the parameter space are defined by the solution of an eigenvalue problem which can be obtained by linearizing the RG equation,

$$\delta h(x) = a\Big(F_0^{N-1}(x) h(x/a) \\ + \sum_{m=1}^{N-2} F_m^{N-1}(x) h(g^m(x/a)) + h(g^{N-1}(x/a)) \Big), \tag{4}$$

where

$$F_m^{N-1}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}\xi} [g^{N-m-1}(\xi)] \right)_{\xi = g^{m+1}(x/a)},$$

$$N = 2^k, \quad k = 1, 2, 3, \dots .$$

For each critical point there exist two essential eigenvalues δ_1 and δ_2 which are greater than unity in modulus and are not connected with infinitesimal variable changes. These constants are the universal scaling factors. Choosing the appropriate coordinate system in the parameter plane (*scaling coordinates*), the structure of the domains of different dynamical regimes is reproduced under rescaling along the axes by the factors δ_1 and δ_2. Respectively, the characteristic time of dynamical regimes is rescaled by a factor 2^k.

The critical points of codimension two (as well as common Feigenbaum period doubling accumulation points) possess the following property: at these points the map has all possible 2^n-period cycles. All these cycles are unstable: a small perturbation Δx appears to be $\mu \Delta x$ after a one-cycle period, and $|\mu| > 1$. μ is called a *multiplier*. We emphasize *the universality of critical multipliers*. In the case of periodic UD codes this can be formulated in the following way.

At the codimension-two critical point having a k-period UD code the multipliers of the 2^n-period cycles depend periodically on n. They take universal values periodically repeated in the proper order (defined by the code type only), μ_i, $i=1, \dots, k$.

The values μ_i may be calculated with high precision if the appropriate solution of the functional equation (3) is found.

Table 1 presents the universal constants a, δ, μ_i for points with some simple UD codes (see also ref. [8]) [#1].

3. With the results of the previous section, we can

For footnote see next page.

Table 1
Universal constants for some codimension-two critical points.

Code	a	δ	μ_n
UUUUUU..., period 1	−1.69030297	7.28768622	−2.0509405
		2.85712414	
DDDDDD..., period 1	2.85712414	7.28768622	−2.0509405
		2.85712414	
UDUDUD..., period 2	−4.86264509	35.9286114	−2.2751695
		14.5957450	−2.2751695
UUDUUD..., period 3	8.03026759	244.768707	−2.1434758
		46.2910330	−2.2539228
			−2.2778750
UUDDUU..., period 4	23.61530584	1275.15727	−2.1663709
		195.693975	−2.2407195
			−2.1663709
			−2.2407195

Table 2
Codimension-two critical points for map (5).

Code	D	A	B
UUUUUUUUU...	0	−0.242698757265	1.951385777782
	0.1	–	–
	0.2	–	–
	0.3	–	–
UDUDUDUDU...	0	−0.158717925945	2.102336520597
	0.1	−0.168054744666	2.217010484297
	0.2	−0.168801835391	2.335895960921
	0.3	−0.184910239985	2.464150590153
UDUUDUUDU...	0	−0.160653611834	2.097746013474
	0.1	−0.169536315710	2.212343535005
	0.2	−0.177926918535	2.333635384400
	0.3	−0.185731259347	2.459482435628
UDDUUDDUU...	0	−0.150462468665	2.093181820358
	0.1	−0.160001191578	2.211781560499
	0.2	−0.168973681291	2.335465492391
	0.3	−0.177349557819	2.463397230676

try to find codimension-two critical points for a particular two-dimensional Hénon-like map, for example,

$$x_{n+1}=A-Bx_n+x_n^3-Dy_n, \quad y_{n+1}=x_n\,. \qquad (5)$$

When $D=0$ it is not difficult to calculate the coordinates of a codimension-two critical point with the desirable accuracy by estimating the limit point of the sequence of the double superstable cycles (see table 2). However, when $D\neq 0$ we cannot easily reproduce this method of searching for the critical point because it is difficult (if possible at all) to outline the points in the phase space of the two-dimensional map which play a role analogous to the extrema of

[#1] The case of UD codes reproducing themselves under the shift and symbol exchange U↔D requires a special discussion. For such codes the parameter space topography and the values of the critical multipliers reproduce themselves not only after k period doublings, but also after $\frac{1}{2}k$. This fact is connected with the actual equality of the role played by the two extrema of the bimodal map. The theory of MacKay and van Zeijts [8] takes this fact as one of the basic ideas, but this is not explicitly evident in the simplified version of the RG analysis used here.

the one-dimensional map. Hence we shall turn to an alternative approach.

For a critical point with a certain periodic UD code, we first find the corresponding values of the critical multipliers. Then, let us increase the value of the parameter D from zero and trace the 2^n- and 2^{n+1}-period cycles of the two-dimensional map. Their multipliers are obtained as eigenvalues of the Jacobian matrices,

$$\mathfrak{J}_n = \begin{pmatrix} f'(x_{2^n}) & -D \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f'(x_2) & -D \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f'(x_1) & -D \\ 1 & 0 \end{pmatrix} \quad (6)$$

and $\mathfrak{J}_{n+1}$. During the process of tracing we use the Newton–Raphson technique and choose the values of A and B in such a way that the multipliers the greatest in modulus remain equal to the universal numbers μ_n and μ_{n+1}, respectively. The results should converge to a definite limit which increasing n. The critical point for some non-zero parameter D being found, the existence of the scaling properties which are characteristic for the universality class can be verified.

It should be noted that the above developed method does not hold when we try to apply it to tricritical points. The reasons for this were discussed, in particular, in ref. [11]. We have discovered, however, that the method holds for other types of criticality. In table 2 we present the coordinates of critical points with UD codes of period 2, 3, 4 obtained for the two-dimensional map (5).

4. Our goal is now to ascertain that the critical situations found can be related to the same classes of universality found in the RG analysis of bimodal one-dimensional maps.

Figures 1–3 demonstrate local scaling in the phase space which is characteristic for the different critical points. Part a of these figures shows the complete attractors, and parts b–d show the fragments of part a at several subsequent steps of rescaling. The magnification factors at each step were equal to the appropriate constants a found from the RG analysis (see table 1). We expect that the figures obtained at high levels of magnification reproduce the previous ones. Figures 1–3 show that this is really so.

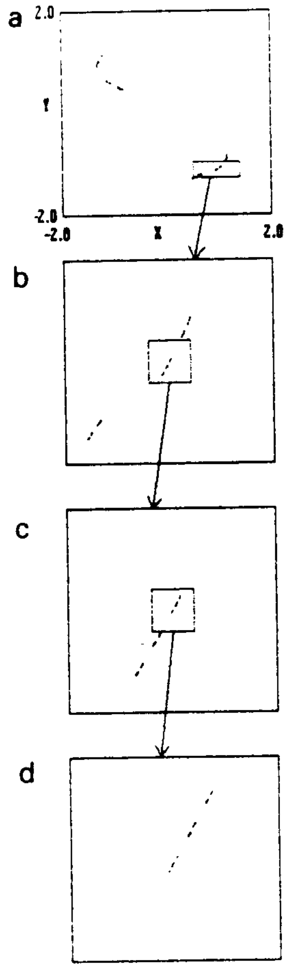

Fig. 1. Demonstration of the local scaling at the attractor for the two-dimensional map (5) at the critical point with the code UDUDUD.... The parameter values are $A=-0.184910239985$, $B=2.464150590153$, $D=0.3$, and the rescaling factor $a=4.8626...$.

In fig. 4 power spectra are compared for the time series generated by the one- and two-dimensional maps at critical points of the same type. It should be noted, that each critical situation possesses its own type of spectrum with inherent relations between the subharmonic amplitudes of different levels. Figure 4 demonstrates a good agreement between spectra generated by one- and two-dimensional maps at critical points of each definite type. This is one more convincing proof that their dynamics refer to the same universality classes.

Finally, we must verify the scaling properties of the parameter space in the neighborhoods of the critical points. Let us calculate the derivatives of the multipliers with respect to the parameters A and B for cycles of period 2^n, 2^{n+k} and 2^{n+2k}, and construct the matrices

$$\mathbf{M}_n=\begin{pmatrix}\partial\mu_n/\partial A & \partial\mu_n/\partial B\\ \partial\mu_{n+k}/\partial A & \partial\mu_{n+k}/\partial B\end{pmatrix},$$

$$\mathbf{M}_{n+k}=\begin{pmatrix}\partial\mu_{n+k}/\partial A & \partial\mu_{n+k}/\partial B\\ \partial\mu_{n+2k}/\partial A & \partial\mu_{n+2k}/\partial B\end{pmatrix}. \quad (7)$$

Here k is a period of the UD code. If the perturbation vector $\boldsymbol{r}=(\Delta A, \Delta B)$ corresponds to the eigendirection referring to the eigenvalue δ_i (the direction of the ith scaling coordinate axis) then we would have

$$\hat{\mathbf{M}}_{n+k}\boldsymbol{r}=\delta_i\hat{\mathbf{M}}_n\boldsymbol{r}. \quad (8)$$

Thus, the factors δ_i would be obtained as the eigenvalues of the matrices $\hat{\mathbf{M}}_n^{-1}\hat{\mathbf{M}}_{n+k}$ (more accurately, this holds only in the asymptotic case of large n, when the scaling property is rigorously valid). Table 3 contains the eigenvalues calculated via such a procedure for the critical points having simple UD codes of periods 2, 3 and 4. One can see that they are in good agreement with the data of table 1 found with the help of the RG analysis.

The above described procedure permits one to estimate the eigendirections of scaling as well, they can be obtained as eigendirections of the matrices

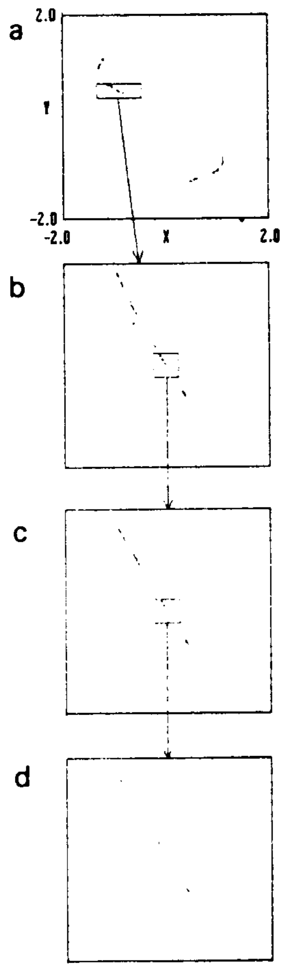

Fig. 2. Demonstration of the local scaling at the attractor for the two-dimensional map (5) at the critical point with the code UDUUDUUD.... The parameter values are $A=-0.185731259347$, $B=2.459482435628$, $D=0.3$, and the rescaling factor $a=8.0302$....

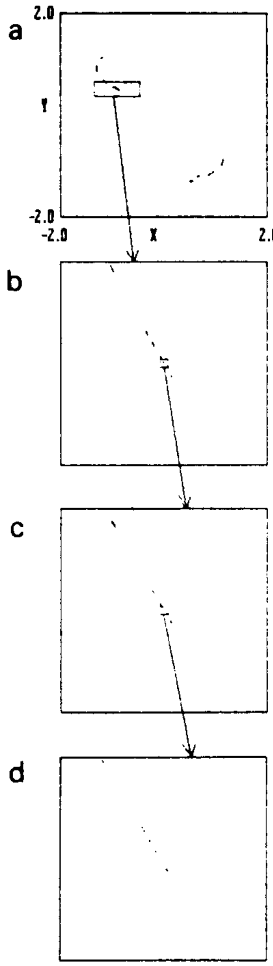

Fig. 3. Demonstration of the local scaling at the attractor for the two-dimensional map (5) at the critical point with the code UDDUUDDUUDD.... The parameter values are $A=-0.177349557819$, $B=2.463397230676$, $D=0.3$, and the rescaling factor $a=23.6153$....

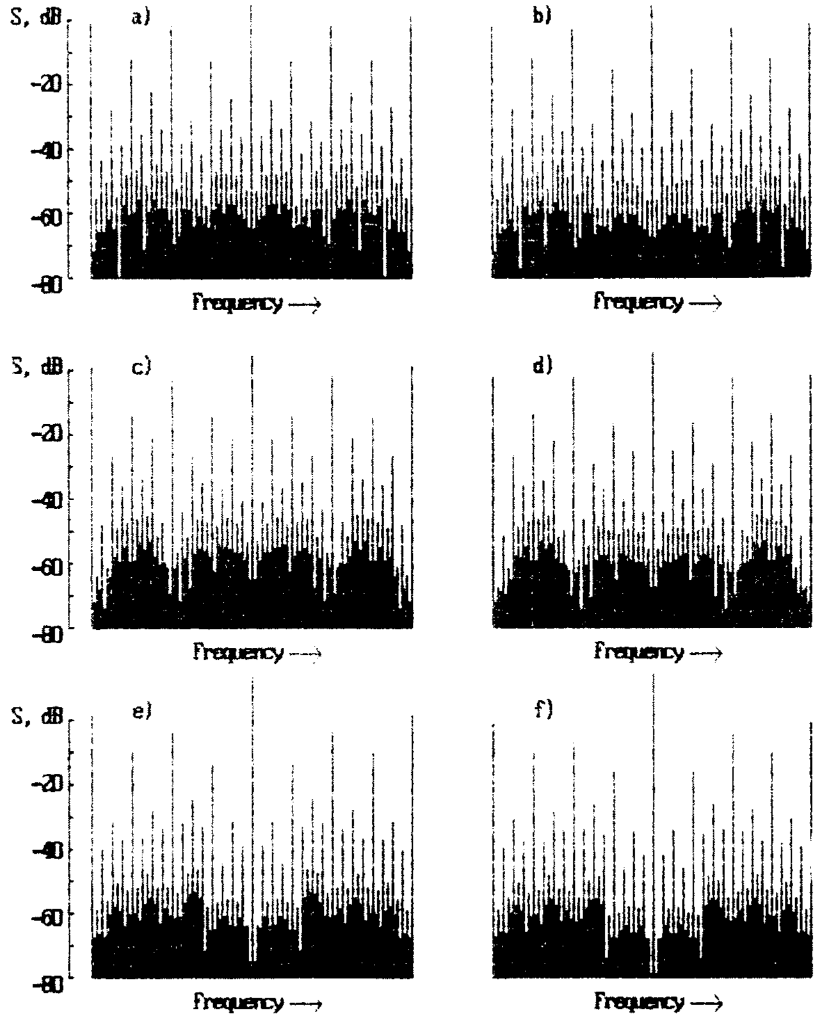

Fig. 4. Comparison of the Fourier spectra generated by: (a), (c), (e) the one-dimensional map (eq. (5) for $D=0$) and (b), (d), (f) the two-dimensional map (eq. (5) for $D=0.3$) at the similar critical points of codimension two. Codes: (a), (b) UDUDUD... ; (c), (d) UDUUDUU...; (e), (f) UDDUUDDUU.... See the parameters A and B in table 2.

$\hat{\mathbf{M}}_n^{-1}\hat{\mathbf{M}}_{n+k}$. Making the calculations for the critical points from table 2 at $D=0.3$ and designating the scaling coordinates as ξ and η we obtain:

(i) code UDUDUDUD...

$$A=-0.1849102+0.819\xi-0.095\eta\,,$$

$$B=2.4641506+\xi+\eta\,, \tag{9}$$

(ii) code UDUUDUUD...

$$A=-0.1857312+\xi-0.04\eta\,,$$

$$B=2.4594824-0.32\xi+\eta\,, \tag{10}$$

(iii) code UDDUUDDU...

$$A=-0.1773496+\xi-0.155\eta\,,$$

$$B=2.4633972+0.08\xi+\eta\,. \tag{11}$$

Figures 5–7 show the numerically obtained topography of the (A, B) parameter plane for map (5) near three codimension-two critical points. These figures make use of the scaling coordinated systems. In each figure a fragment is selected and shown separately magnified by factors δ_1 and δ_2 along two axes. Here δ_1 and δ_2 are the respective scaling factors presented in table 2. We see that the parameter plane topography in the selected regions reproduces the whole

Table 3
Comparison of the empirical parameter space scaling factors δ_1 and δ_2 with their exact values from the RG analysis.

Code	Cycles used	δ_1	δ_2
UDUDUDUD...	4–16–64	35.093	17.585
	8–32–128	36.582	13.784
	16–64–256	35.465	14.934
	32–128–512	36.016	14.551
	64–256–1024	35.930	14.590
	128–512–2048	35.928	14.600
	RG result	35.929	14.596
UDUUDUUD...	4–32–256	236.491	52.801
	8–64–512	247.697	44.693
	16–128–1024	247.312	45.256
	32–256–2048	244.659	45.899
	64–512–4096	244.839	46.298
	RG result	244.769	46.291
UDDUUDDUU...	2–32–512	1261.458	207.753
	4–64–1024	1215.913	210.063
	8–128–2048	1262.414	203.532
	16–256–4096	1263.882	194.787
	RG result	1275.157	195.694

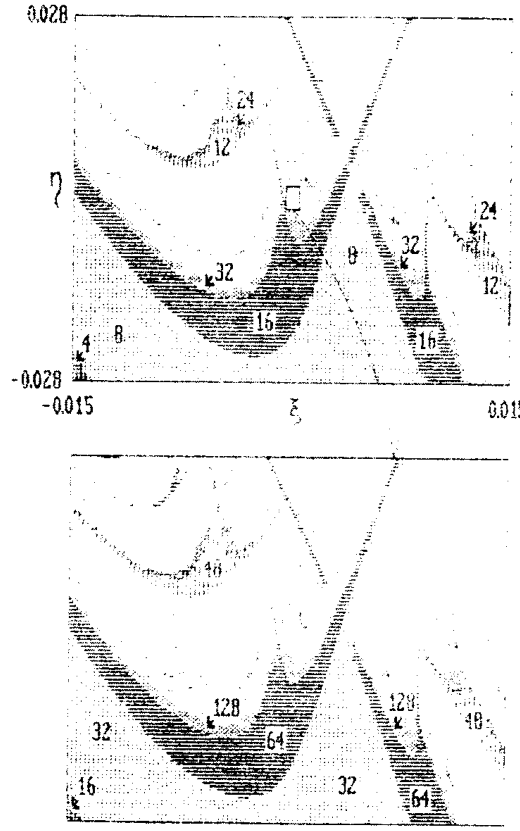

Fig. 5. Demonstration of the scaling properties for the parameter space of map (5), $D=0.3$. The topography of the dynamical behavior is shown in scaling coordinates in the neighborhood of the critical point generated by the code UDUDUD... . The periods of the stable cycles are denoted by the corresponding numbers. The critical point is located exactly at the middle of the diagram. A selected box is shown separately under magnification by 35.928... and 14.595... along the horizontal and vertical axes, respectively.

initial picture under such rescaling. It supports once more the assertion that the two-dimensional Hénon-like map behavior belongs to the universality classes revealed by the RG analysis of one-dimensional bimodal maps.

5. The above results give us reasons to suppose that most of the universal quantitative laws revealed for the transition to chaos via period doublings in one-dimensional bimodal maps remain valid for two-dimensional Hénon-like maps as well. This concerns not only Feigenbaum's segments of the chaos boundary, but the codimension-two critical points too, *with the exception of tricritical points* which have a UD code in the form of only one infinitely reported symbol U or D. The reason for such a difference between tricriticality and other codimension-two critical points may be explained by the following simple heuristic arguments.

Suppose, we want to realize some definite codimension-two critical situation in the Hénon-like map F (see (1)). A map defining the dynamics over the discrete time interval of 2^n iterations can be presented as a composite of two maps F^p and F^q where the pair (p, q) is the nth term of the sequence given by rule (2). Since the Jacobian of the critical map F equals D, then the Jacobians of the maps F^p and F^q are D^p and D^q, respectively. Because $D<1$, both Jacobians will tend to zero with $n\to\infty$ if both numbers p and q tend to infinity. This means that both F^p and F^q will tend to one-dimensional maps, and this is the case when the theory of one-dimensional bimodal maps describes adequately the critical behavior in the asymptotic case of large n. The above condition is, apparently, satisfied for all infinite UD codes except the tricritical ones.

On the contrary, in the tricritical case, when we have the repeated symbol U (or D) in the code, the

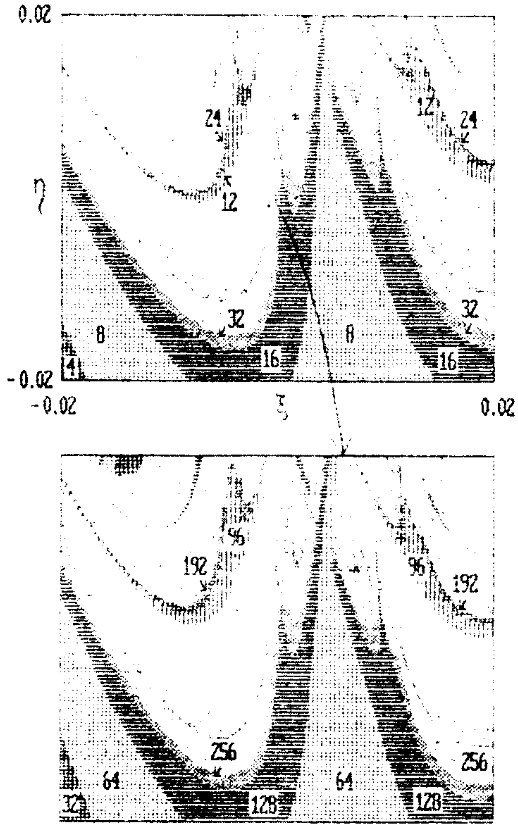

Fig. 6. Demonstration of the scaling properties for the parameter space of map (5), $D=0.3$. The topography of the dynamical behavior is shown in scaling coordinates in the neighborhood of the critical point generated by the code UDUUDUUD… . The periods of the stable cycles are denoted by the corresponding numbers. The critical point is located exactly at the middle of the diagram. A selected box is shown separately under magnification by 244.76… and 46.292… along the horizontal and vertical axes, respectively.

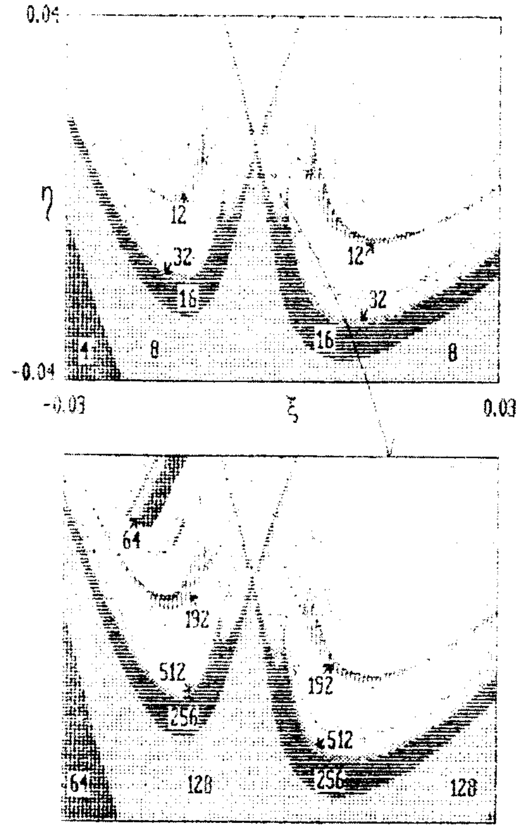

Fig. 7. Demonstration of the scaling properties for the parameter space of map (5), $D=0.3$. The topography of the dynamical behavior is shown in scaling coordinates in the neighborhood of the critical point generated by the code UDDUUDDU… . The periods of the stable cycles are denoted by the corresponding numbers. The critical point is located exactly at the middle of the picture. A selected box is shown separately under magnification by 1275.15… and 195.693… along the horizontal and vertical axes, respectively.

numbers p_n (or q_n) will remain constant with increasing n. This means that one of the two maps F^p (or F^q) retains in the $n\to\infty$ limit a finite value of the Jacobian and does not reduce to a one-dimensional map. Let us demonstrate the consequences of this statement. Suppose we have for $D=0$ a situation where the extremum is mapped to the extremum. So the composition F^pF^q has a quartic extremum, and tricriticality is realized at the limit of the period doubling cascade under this condition. However, one can see that for $D>0$ a quartic extremum does not exist. To show it more clearly, let us consider an example – the composite of the maps F^p: $(x, y)\to(x^2-Dy, x)$ and F^q: $(x, y)\to(x^2, x)$. The resultant map F^pF^q: $(x, y)\to(x^4-Dx, x)$ is equivalent to a one-dimensional map $x\to x^4-Dx$ but the quartic extremum exists at $D=0$ only.

We conclude that the picture of transition to chaos in two-dimensional Hénon-like maps cannot be imagined as a trivial generalization of the one-dimensional bimodal map picture. Some details are maintained, and some details do not hold (tricriticality). Thus, the problem of global arrangement of the chaos boundary in a parameter plane demands further careful investigations. Here we have shown some important peculiarities concerning the universal behavior at the onset of chaos. We believe that these peculiarities are valid for a wide class of multidimensional dissipative systems. The developed methodology can be useful also for understanding the behavior of multidimensional generalizations of a circle map with two inflection points [12].

This work was supported in part by a Soros Foundation Grant award by the American Physical Society.

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