Fractals in the Natural and Applied Sciences (A-41)
M.M. Novak (Editor)
Elsevier Science B.V. (North-Holland)

MULTI-PARAMETER TRANSITION TO CHAOS AND FRACTAL NATURE OF CRITICAL ATTRACTORS

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Fractal properties are discussed for critical attractors at the onset of chaos in multi-parameter analysis of one and two-dimensional maps. Each of the critical situations is characterized by a kind of 'visiting card' containing distinctive scaling constants, generalized dimensions, f(α)-spectra, Fourier spectra and other quantifiers.

Keyword Codes: F.1.3, G.3, H1.0
Keywords: Complexity Classes; Probability and Statistics; Models and Principals

1. INTRODUCTION

Feigenbaum's attractor arising at the limit point of the period-doubling cascade is one of classic objects for testing different technique to operate with multifractal sets[1-4]. However, this is only the simplest example of a multifractal attractor. In this report we describe other kinds of such attractors included in one- and two-dimensional state space. We hope that consideration of these objects will be interesting both for further developing multifractal formalism and understanding nonlinear dynamics near the onset of chaos. Various examples of multifractal attractors appear when we attempt to adopt an idea of scenario of transition to chaos for a case of several control parameters.

The commonly recognized research program of the bifurcation and catastrophe theories traced back to Poincare, is based on considering phenomena in the order of increasing codimension. The codimension is the minimal number of parameters typifying a phenomenon. Analogous approach to a problem of transition to chaos we call α *theory of multi-parameter criticality*, because many people speak about critical phenomena in nonlinear systems having in mind dynamics at the onset of chaos. Attractors arising in such situations we call *critical attractors.*

2. CRITICAL ATTRACTORS OF ONE-DIMENSIONAL MAPS

Let us begin with one-parameter families of one-dimensional maps $x \rightarrow f(x)$, which exhibit transition to chaos via the period-doubling cascade. It is well known that Feigenbaum's universality is valid in this case[5,6]. The simplest representative of this universality class is the logistic map having one quadratic extremum (see Fig. 1a).

$$x_{n+1} = 1 - \lambda x^2_n \quad (1)$$

The threshold of chaos corresponds to the limit point of the period-doubling bifurcations. $\lambda = 1.40115518909$. This is *Feigenbaum's critical point* which we denote by F.

In two-parameter families of one-dimensional maps with two quadratic extrema, there is a possibility to find a curve γ in the parameter plane, which is defined by a condition that one extremum is mapped into another after one iteration (Fig. 1b). Evidently, at this curve the two-fold iterated map $x \rightarrow f(f(x))$ has an extremum of the forth order. So, the period-doubling cascade (if it is observed while moving along the γ curve) obeys a law specific for the map having a quartic extremum rather than a quadratic one. A limit point of this cascade at the boundary of chaos is called *tricritical point* (denoted by symbol T)[7]. These points appear as end points of Feigenbaum's critical lines. For example, in the map

$$x_{n+1} = A - Bx_n + x_n^3 \quad (2)$$

The γ curve is given by the equation $A=(B/3)^{1/2}(1-2B/3)$, and the tricritical point located at this curve has the coordinates $A_T=-0.242698757265$, $B_T=1.951385777782$.

At last, let us turn to three-parameter families of one-dimensional maps $x \rightarrow f(x)$. Then we can single out four distinctive situations, which appear at some curve lines in three dimensional parameter space:

(i) At the extremum point the function f(x) has zero derivatives of the second and third order (Fig. 1c).

(ii) Function f(x) has both a quadratic extremum and a cubic inflection point, and the quadratic extremum is mapped to the inflection point (Fig. 1d).

(iii) Function f(x) has both a quadratic extremum and a cubic inflection point, and the inflection point is mapped to the quadratic extremum (Fig. 1e).

(iv) Function f(x) has three quadratic extrema, the first extremum being mapped exactly to the second one and that, in turn, to the third one (Fig. 1f).

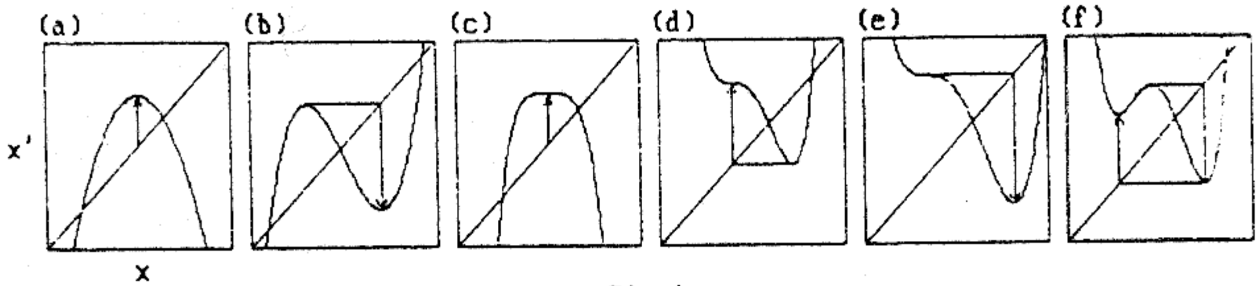

Fig.1

The simplest map exhibiting all the situations (i)-(iv) is

$$x_{n+1} = 1 - Ax^2_n - Bx^4_n - Cx_n. \quad (3)$$

At the curve (i), the function f(x) has a quartic extremum. At the curves (ii) and (iii) the second iteration f(f(x)) has an extremum of the 6th order. At the curve (iv) the third iteration f(f(f(x))) has an extremum of the 8th order. Thus, if we move along one of the curves (i) -(iv) and observe a period-doubling cascade, the law of its convergency is the same as in the map $x \to 1 - \lambda |x|^m$, m=4, 6, 6, 8, respectively. In the case (i) the period-doubling cascade converges to the T-point, but this variant of tricriticality demands three parameters for its realization. We introduce symbols S, S', and E for critical behaviour at the onset of chaos in situations (ii), (iii), and (iv), respectively. The notation stands for the first letters of the numbers "six" and "eight". For the map (3) we find the critical points

T: A=0, B=1.5949013562288, C=0
S: A=1.872448192264, B=-1.625205284712, C=1.094016101529
S': A=1.379909480783, B=-0.557409701182, C=1.181821122325
E: A=2.449366934076, B=-1.260415730596, C=0.700954625016.

Each type of criticality F, T, S, S', and E has its own quantitative universality and scaling properties. It follows from the fact that a form of the 2^k-fold iterated map is given by a universal function g(x), if we use some normalization of the x variable. The g(x) functions are fixed points of the Feigenbaum's renormalization group (RG) equation

$$g_{n+1}(x) = \alpha_n g_n(g_n(x/\alpha_n)), \quad (4)$$

where α_n is a scaling constant fixed by a normalization condition, $\alpha_n = 1/g_n(1)$). A cause of the universality consists in a possibility to find the g(x) functions as the fixed point solutions of the RG equation without appellation to any initial map f(x).

The RG equation fixed point corresponding to the typical one-parameter period-doubling critical behaviour F was obtained by Feigenbaum [5,6]. This function g(x) is presented by polynomial approximation including powers of x^2. The corresponding scaling constant is $\alpha_F = -2.502907876$.

To uncover the universality and scaling properties intrinsic to the two- and three-parameter criticality, we need to turn to a generalization of Feigenbaum's theory for the maps with non-quadratic extrema [8,9]. If the extremum order is m then the corresponding solution of Eq. (4) contains the terms of the $|x|^m$, $|x|^{2m}$, $|x|^{3m}$,... In Refs. [8,9] the dependence of the solutions on m was investigated, while m was any real number. For m=4, 6, and 8 one can find $\alpha_T = -1.6903029714$, $\alpha_S = -1.4677424503$, and $\alpha_E = -1.35801728$.

Each of the above critical situations give rise to a kind of Cantor-like attractor. Local property of self-similarity consists in reproducing the attractor structure under scale change by the factor of α near appropriate point in the state space. For Feigenbaum's case an explicit procedure is known to construct this set. At first, find a sequence x_i generated by a map at the Feigenbaum's point, starting from the extremum point $x_0 = 0$. At the levels number k=0, 1, 2, ... the attractor is approximated simply by a unification of 2^k segments:

$$A_k = \bigcup_{i=1}^{2^k} [x_i, x_{i+2^k}]. \quad (5)$$

The attractor itself is obtained as a limit object for $k \to \infty$. Such a construction can be adopted easily for the critical attractors of T, S, S', and E types.

Using the above procedure and technique of Ref.1 we can calculate the basic characteristics of the critical attractors as multifractal sets, f(α)-spectrum and generalized dimensions. Let us take some definite k-th level of the Cantor-like construction, when the attractor is approximated by a set of 2^k segments of lengths l_i, and probabilistic measures $p_i = 1/2^k$ are attributed to the segments. Let us define a partition function depending on two parameters q and τ

$$\Gamma_k (q, \tau) = \sum_{i=1}^{2^k} p_i^q / l_i^\tau \tag{6}$$

and require

$$T_k (q, \tau) \rightarrow \text{const}, \ k \rightarrow \infty. \tag{7}$$

Due to the last condition we obtain a relation between q and τ, q = q(τ). Then we have

$$\alpha = (dq/d\tau)^{-1}, \quad f = \alpha q - \tau, \quad D = \tau/(q-1). \tag{8}$$

Changing τ as a parameter we find two functions f(α) and D(q) from (8), which give us the f(α)-spectrum and the spectrum of generalized dimensions, respectively (see Figs. 2 and 3 Note that the plots f(α) and D(q) for two- and three-parameter tricriticality do not differ, and for S and S' types they also coincide.

One more useful characteristic of critical dynamics is an ordinary Fourier spectrum These spectra are presented in Fig.4 (a - Feigenbaum's point, b - tricritical point, c- S-point and d - E-point). They exhibit an infinite number of subharmonics with frequencies ω proportional to 2^{-k} and have a hierarchical organization: each k-th subharmonic level has less amplitude than the previous one. However, the quantitative relations between the levels are specific for each type of criticality.

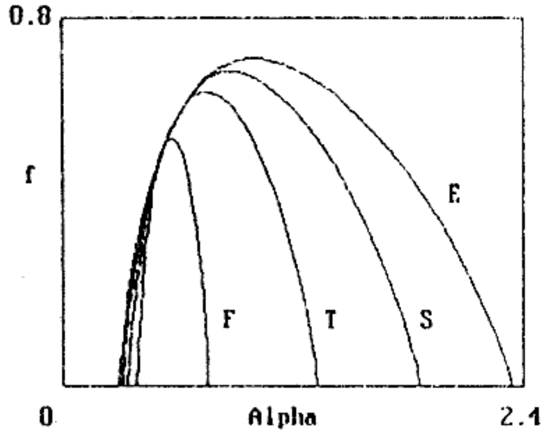

Fig.2

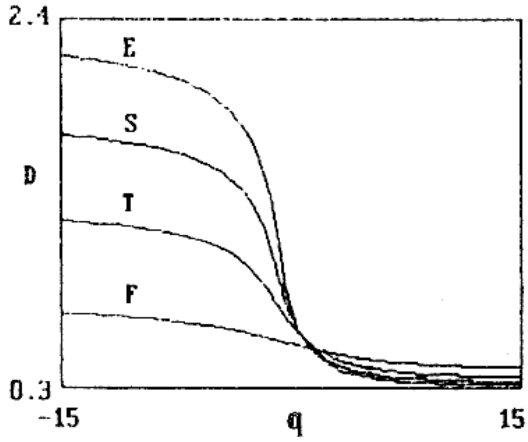

Fig.3

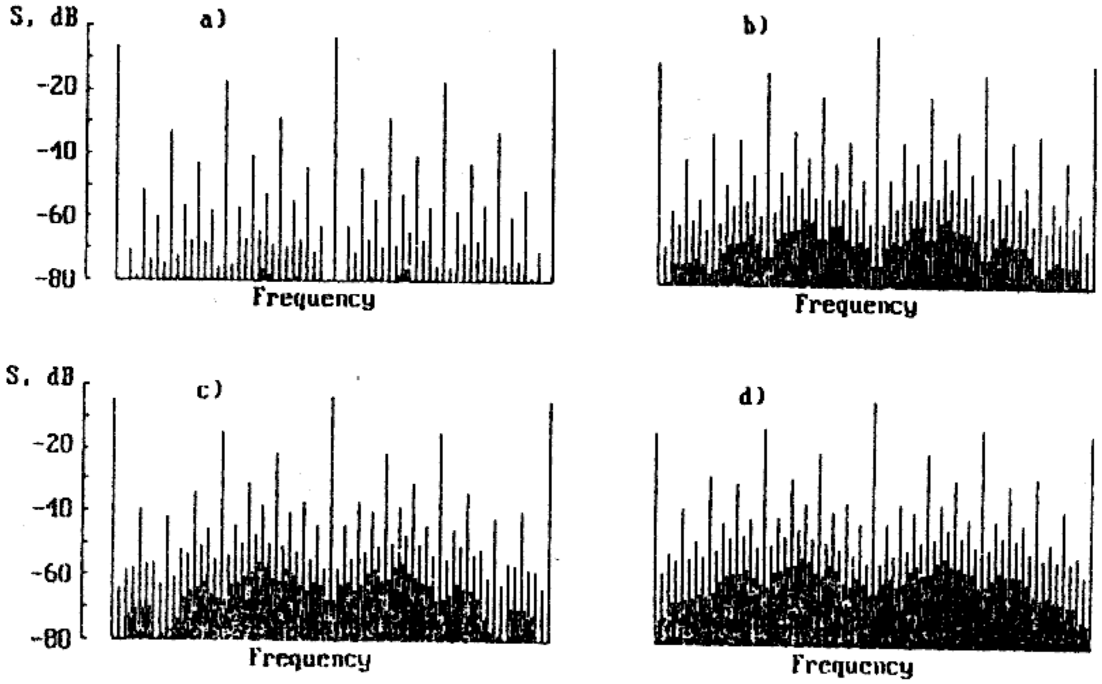

Fig.4

A set of distinctive quantitative characteristics including universal functions, scaling factors, f(α), D(q), and Fourier spectra may be considered as a "visiting card" or "identification" of each criticality type. Due to universality, any system of arbitrary physical nature will present the same "visiting card", if the definite type of criticality occurs.

Further analysis of two- and three-parameter families of one-dimensional maps shows that the above considered types of criticality do not exhaust all possibilities of behaviour at the onset of chaos via period-doubling cascades. Due to Kapral, MacKay and other authors [10-14] we know that there exhists an infinite Cantor-like set of codimension-two critical points in a parameter plane of one-dimensional map having two extrema. These points belong to the boundary of chaos and appear as limit points of period-doubling cascades along definite paths in the parameter plane. A set of such paths forms a binary tree, and each critical point is coded by an infinite sequence of two symbols U and D (in other notation, R and L [10-13]).

Coordinates of each codimension-two critical point can be found as a limit of a sequence of double super-stable cycles (the cycles having both extrema of the considering map among their elements). If the maximum is mapped into the minimum after p iterations, and the minimum is mapped into maximum after q iterations, we speak about (p,q)-type cycle.

To construct the sequence (p_k,q_k), which leads to the critical point with a definite UD-code, we start from the cycle of the (1,1)-type. Then we take subsequent symbols of the UD-code and calculate p and q numbers by the relations

$$(p_{i+1}, q_{i+1}) = \begin{cases} (p_i, p_i+2q_i), & \text{if U,} \\ (2p_i+q_i, q_i), & \text{if D.} \end{cases} \qquad (9)$$

The type of critical behaviour appears to depend on a structure of the UD-code. In particular, the codes with tails ...UUUU... and ...DDDD... correspond to tricritical points. If the tail contains a periodically repeating set of M symbols then the dynamics at the critical point is described by a period-M solution of the RG equation (4). For such a solution we define scaling factor "over the RG-cycle period" as a product of M values of α_k. For example, for simple codes of period 2, 3, and 4 we have (see also [13, 14]):

UDUDUDUD...: $\alpha = \alpha_k\alpha_{k+1} = -4.8626450906$,
UUDUUDUU...: $\alpha = \alpha_k\alpha_{k+1}\alpha_{k+2} = 8.0302675872$,
UUDDUUDD...: $\alpha = \alpha_k\alpha_{k+1}\alpha_{k+2}\alpha_{k+3} = 23.6153058715$.

For the model map (2) we find the following coordinates of critical points with these codes:

UDUDUDUD..., A = -0.1587179259453, B = 2.102336520597,
UUDUUDUU..., A = -0.2211510892692, B = 2.016490507000,
UUDDUUDD..., A = -0.2193259886681, B = 2.017904888546.

Critical attractors at these points also exhibit fine Cantor-like structure. However, the explicit procedure of their construction is more sophisticated. Let us find two sequences y_i and z_i generated by considering a map $x \rightarrow f(x)$:

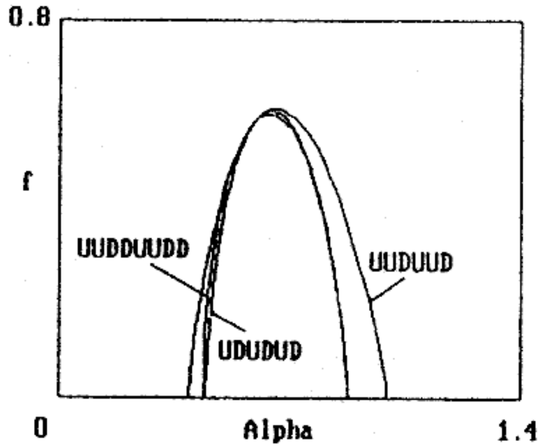

Fig.5

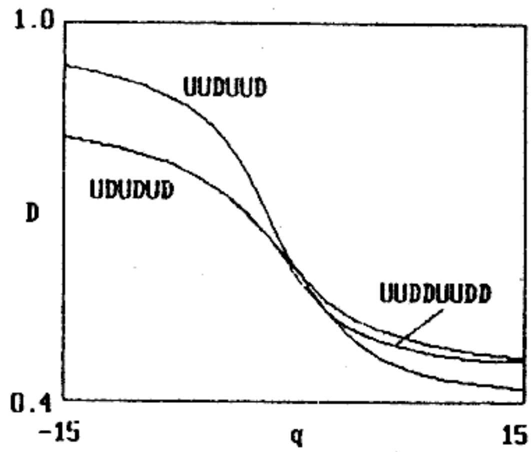

Fig.6

The first starts from the point of the maximum (y_0) and the second starts from the minimum (z_0). At the k-th level of the construction we recall the k-th pair (p,q) from the sequence of double superstable cycles converging to the critical point under consideration. Then we take p terms of the y_i sequence and q terms of the z_i sequence and define

$$x_i = \begin{cases} y_i, & 1 \le i \le p, \\ z_{i-p}, & p < i \le p+q. \end{cases} \qquad (10)$$

The numbers x_i give ends of segments for the attractor approximating at the k-th level (see (5)). The attractor itself is obtained in the limit $k \rightarrow \infty$. Now we can evaluate $f(\alpha)$-spectra and D(q)-dependence for critical points having different UD-codes, see Figs.5 and 6. (Note that dealing with period-M cycle of the RG equation, it is reasonable to use a condition $\Gamma_{n+M} = \Gamma_n$ instead of (7) to obtain a fast convergency.) Fig. 7 shows Fourier spectra.

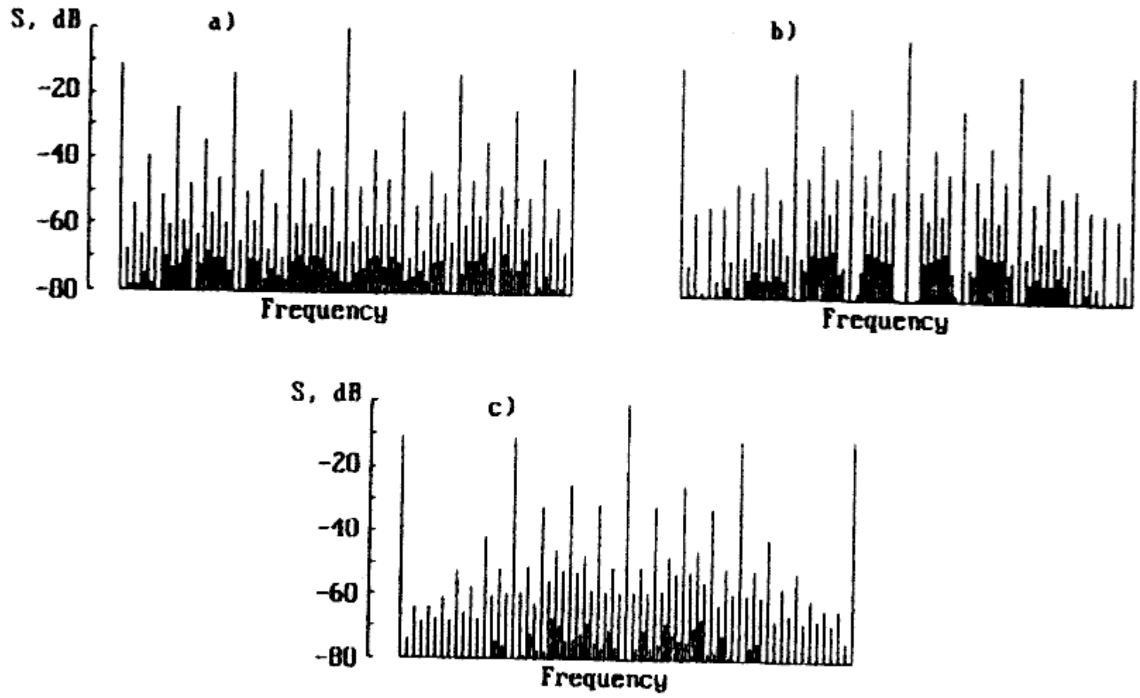

Fig.7

In a similar manner, an infinite number of criticality types coding by symbolic sequences can be found also in three-parameter case; then the simplest period-1 codes correspond to S or E critical points.

3. CRITICAL ATTRACTORS IN TWO-DIMENSIONAL DISSIPATIVE MAPS

In this section we discuss examples of critical attractors included into two-dimensional state space. We demonstrate impressive properties of local self-similarity, giving evidence of their fractal nature. However, these attractors are more complicated than the above considered ones, and further development of multifractal formalism seems to be needed for their complete description.

We model two-dimensional map corresponding to the case of unidirectional coupling of two subsystems:

$$x_{n+1} = 1-\lambda x^2_n, \qquad y_{n+1} = 1- Ay^2_n - Bx^2_n, \tag{11}$$

where λ, A, B, are parameters, x and y are state variables of the first and the second subsystem, respectively. The first equation does not depend on the second one, and the x component undergoes period-doubling bifurcations at the known parameter values $\lambda_k = 0.75$, 1.25, 1.3680989,... Suppose, we are moving in the (λ, A) parameter plane along a line $\lambda = \lambda_k$. For small A we have one multiplier of the period-2^k cycle equal (-1), and the second one near zero. Increasing A we come to the point where the second multiplier also becomes equal to (-1). It means that a new mode has come onto the boundary of stability. The found point (λ_k, A_k) we call the terminal point. Then we repeat the procedure for the next k and so on. The limit of the terminal point sequence in the case under consideration has been called bicritical point [15,16]. We denote this point by the symbol B. For particular values of B = 0.375 we find the bicritical point to have the coordinates: $\lambda_B = 1.40115518909$, $A_B = 1.124981403$. Now let us add a quadratic term describing a backward influence of the second subsystem on the first one:

$$x_{n+1} = 1-\lambda x^2_n - Cy^2_n, \quad y_{n+1} = 1- Ay^2_n - Bx^2_n. \tag{12}$$

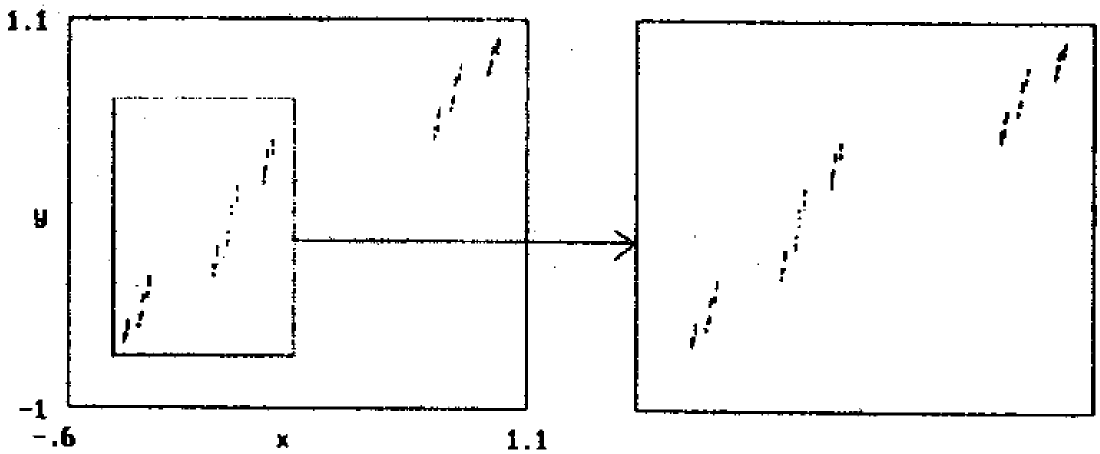

Fig.8

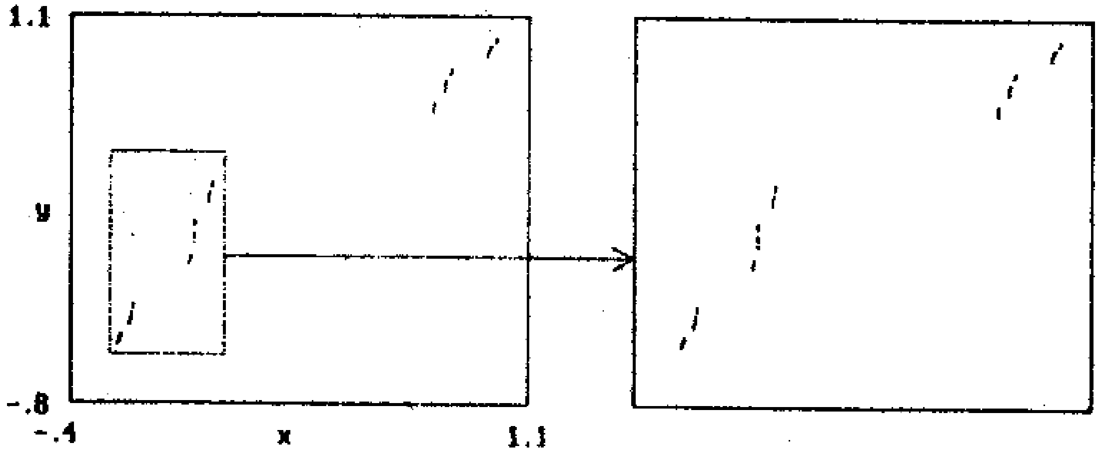

Fig.9

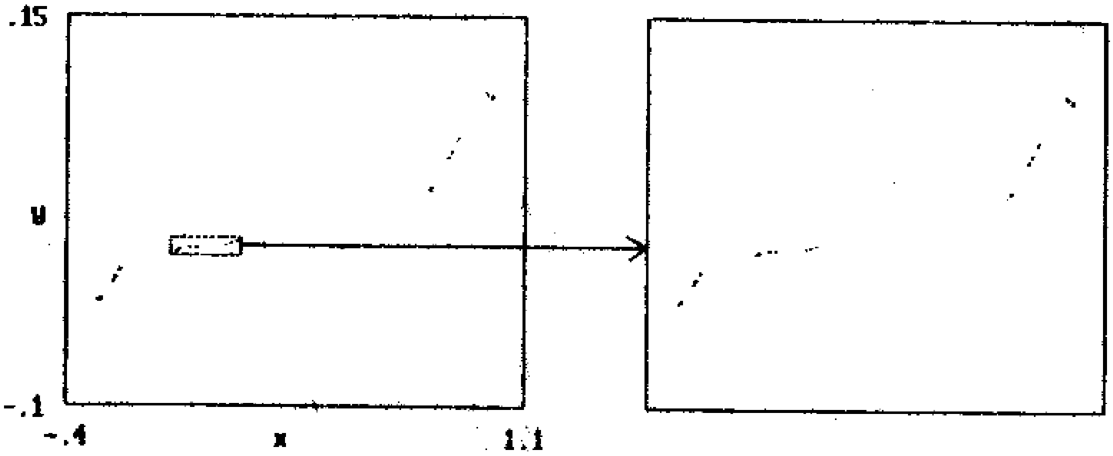

Fig.10

For fixed B and C and small A we observe Feigenbaum's cascade, while increasing λ. We can find the bifurcation values of λ numerically and trace it under increasing A up to the terminal point where the moduli of both multipliers are unity. For moderate negative C we find that a sequence of the terminal points converges to a definite limit. This is the critical point of a new type. One can meet both period-doubling and quasiperiodicity in its neighbourhood, so we denote it as FQ. For B=0.375, C=-0.25 we find λ_{FQ}=1.65452459, A_{FQ}=1.03083759.

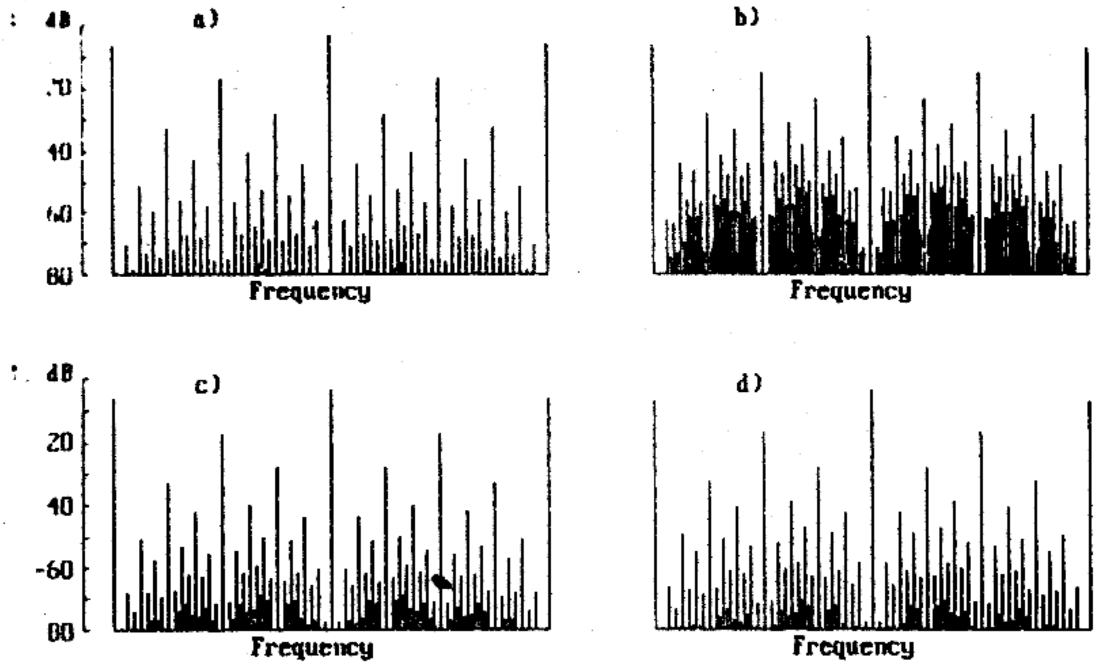

Fig. 11

At last, let us include an odd term:

$$x_{n+1} = 1-\lambda x^2_n - Cy^2_n + \varepsilon x_n, \; y_{n+1} = 1 - Ay^2_n - Bx^2_n. \tag{13}$$

For fixed B = 0.375, C = -0.25, and ε = -0.12 we have reproduced the procedure of movement along bifurcation curves in the (λ, A) parameter plane up to threshold of instability of a new mode. It gives a sequence of terminal points converging to the critical point denoted C. For given values of B, C, and ε we have found λ_c = 1.581493555745, A_c = 1.016156060448.

RG analysis of B, FQ, and C types of criticality is based on two-dimensional generalization of the RG equation (4):

$$\begin{aligned} g_{k+1} &= \alpha g_k(g_k(X/\alpha, Y/b), f_k(X/\alpha, Y/b)), \\ f_{k+1} &= b f_k(g_k(X/\alpha, Y/b), f_k(X/\alpha, Y/b)), \end{aligned} \tag{14}$$

The B and FQ types of criticality are associated with two fixed points of the RG equation [18]. The scaling factors are $\alpha_B = \alpha_F$ = -2.502907876, b_B = -1.505318159 for the bicriticality, and α_{FQ} = -4.008157849, b_{FQ} = -1.900071670 for the FQ-point. The C-type criticality corresponds to a period-2 cycle of the Eq. (14). Scaling factors defined over a period of the RG-cycle are

$\alpha_c = \alpha_k\alpha_{k+1} = 6.565349940$ and $b_c = b_k b_{k+1} = 22.120227422$.

We emphasize that the "scaling variables" X and Y used in eq.(14) do not necessarily coincide with variables of an initial map; in general they are connected via a linear variable change. Taking it into account and having an intention to show the self-similarity of the attrators with extreme clearance, we present, in Figs.8-10, the plots of attractors for the maps being solution of Eq.(14), rather than for particular model maps (11)-(13). We observe that the pictures reproduce themselves under magnification by corresponding factors of α and b along two coordinate axes. We conclude that these attractors have a fractal nature. In the case of bicriticality an explicit procedure of the attractor approximation by sets of rectangles was proposed earlier [16]. However, in other cases the question needs further analysis.

In Fig.11 (a) and (b) we show Fourier spectra generated by x and y components of the map (11) at the bicritical point. Fig.11 (c) and (d) show Fourier spectra generated by maps (12) and (13) at the FQ and C points. In both cases there is no essential difference between spectra of x and y components.

4. CONCLUSION

We have discussed a promising approach to research of transition to chaos which may be called a theory of multiparameter criticality. This is a synthesis of two ideas: (1) an idea of searching for and classification of new phenomena in order of their codimension, and (2) renormalization group analysis. It gives a chance to involve many new examples of multifractal objects (critical attractors) into investigations on multifractal formalism. It seems to be important for its further development.

Acknowledgements

This work was supported by the Russian Fundamental Research Fund under project 93-02-16169. One of the authors (S. P. K.) thanks the Royal Society of Great Britain for financial support to attend the Fractal 93.

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